DESIGN OF NUCLEAR REACTOR MODELS

by

John Campbell MacAlpine III

A Thesis Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
MASTER OF SCIENCE

Major Subject: Nuclear Engineering

Signatures have been redacted for privacy

Iowa State University
Of Science and Technology
Ames, Iowa

1960

TABLE OF CONTENTS

I. INTRODUCTION

Throughout the short history of reactor technology, reactor models in the form of sigma piles and exponential assemblies have been the major devices for evaluating the nuclear properties of various reactor designs without resorting to elaborate full scale experiments. It has become standard procedure to use sigma piles to obtain measurements of cross sections for the various nuclei and to determine diffusing and slowing down lengths for various materials. Either stationary or pulsed neutron sources may be used on the sigma pile with the achievement of equally accurate results; however, the pulsed source is applicable to smaller samples.

In subcritical assemblies design parameters such as the optimum arrangement, spacing and proportions of the fuel, moderator and other materials can be determined. The effects of additions or subtractions of homogeneous and heterogeneous poisons and absorbers such as control rods and reflectors may be examined. Since a subcritical assembly duplicates in detail small portions (5 to 30 percent) of a reactor, quantities such as the Fermi age, disadvantage and thermal utilization factors, diffusion length and migration area can be resolved for specific lattice arrangements.

It is difficult to determine the dynamic operating properties of a critical reactor by conducting tests on a partial reproduction such as a subcritical assembly. For ex-

ample, it is a prodigious task to calculate accurately the differential control rod worth for each control rod for the vast number of relative control rod positions. The solution of the control rod problem would be obtained readily by running tests on a model.

The purpose of this investigation is to develop a method of designing a model reactor which will retain the operational characteristics of the actual reactor. A model of this type could be utilized in the following ways:

1) the simulation of the effects of poison build-up during reactor operation in an unpoisoned model reactor,
2) the determination of control rod worth in a model and the extension of the results to a full scale reactor,
3) the study and optimization of the relationship between excess reactivity, neutron economy and burn-up.

Since the size of a reactor model may be either larger or smaller than the prototype, it is visualized that a number of scale model tests might be performed on reactors which are now in existence. In many cases it would be possible to conduct tests on larger reactors and then utilize the results to design or predict the operating characteristics of a small reactor. Other situations might require testing of small reactors to obtain large reactor data. Thus critical reactor models would add flexibility to the present reactor testing program.

II. REVIEW OF LITERATURE

An investigation of the available literature pertaining to critical and subcritical assemblies has not revealed any previous research effort directed towards the design or construction of a critical reactor model.

As stated previously, models of reactor sections in the form of sigma piles and subcritical assemblies are utilized extensively to obtain reactor design data; however, a review of this literature would be voluminous and of little value to this investigation. Any literature of interest will be referred to at the appropriate points within the context of the investigation and will be enclosed within brackets.

III. REVIEW OF NEUTRON DIFFUSION THEORY

Many detailed equations have been developed to describe precisely the neutron flux pattern in a reactor. Many of these equations apply to specific cases in a rigorous manner requiring many hours of computer time to obtain a solution. The cost and time required to obtain these exact solutions is warranted only for finalizing a given reactor design. For a great number of initial design problems it is most desirable to simplify the theory thus reducing the calculation time.

For the solution of a great variety of reactor problems diffusion theory provides approximate solutions in reasonable time. Elementary considerations of diffusion theory will not be included here since they can be found in many nuclear reactor physics books [1,2,3]. In diffusion theory the rate of change of neutrons in an elemental volume is determined by considering the rate of absorption, leakage and production. The basis equation which governs diffusion theory is

$$\text{production} - \text{absorption} - \text{leakage} = \frac{dn}{dt} \qquad (1)$$

where $\frac{dn}{dt}$ equals the time rate of change of the neutron density.

If it is assumed that the neutrons existing in a reactor are monoenergetic (thermal energy) and that Fick's law of diffusion is applicable, then the neutron leakage per unit volume per second can be represented by

$$\text{neutron leakage per unit volume per second} = -D\nabla^2\phi \qquad (2)$$

where D = diffusion coefficient for flux

∇^2 = symbol used for the laplacian operator

ϕ = neutron flux.

The number of neutrons absorbed per unit volume per second is equal to $\Sigma_a\phi$, where Σ_a is the macroscopic absorption cross section. Since it is assumed that an absorbed neutron is removed from the system, the absorption term in Eq. (1) equals $-\Sigma_a\phi$.

Since it has been assumed that the fission neutrons are produced at thermal energies, the resonance escape probability (p) and the fast fission factor (ϵ) are both equal to unity. Therefore, the overall multiplication factor (K) is equal to the product of η times the thermal utilization (f), where η is the number of neutrons produced per neutron absorbed in the fuel. The production term in Eq. (1) is the product of K times the neutrons absorbed by the fuel ($\Sigma_a\phi$).

By substitution Eq. (1) becomes

$$K\Sigma_a\phi - \Sigma_a\phi + D\nabla^2\phi = \frac{dn}{dt} \qquad (3)$$

For steady-state operation $\frac{dn}{dt}$ equals zero. Thus, Eq. (3) reduces to

$$D\nabla^2\phi - \Sigma_a\phi + K\Sigma_a\phi = 0. \qquad (4)$$

The results obtained from Eq. (4) are admittedly in error due to the assumption that only thermal neutrons exist in the reactor. By multiplying the source term of Eq. (4) by the nonleakage probability for the neutrons during the slowing

down process the fraction of fast neutrons leaking out of the reactor is considered. Since

$$\text{Fast neutron nonleakage factor} = e^{-B_g^2 \tau} \tag{5}$$

where B_g^2 = geometric buckling

τ = neutron age

the incorporation of Eq. (5) in Eq. (4) yields

$$D\nabla^2\phi - \Sigma_a\phi + Ke^{-B_g^2 \tau}\Sigma_a\phi = 0 \tag{6}$$

If the multiplication factor (K) also includes the resonance escape probability and the fast fission factor, both of which were previously equal to unity, additional gains or losses of neutrons during the slowing down process are considered. Equation (6) is actually a modified single group equation which is valid only for regions removed from sources, strong absorbers or boundaries by two or three mean free paths for neutrons.

If Eq. (6) is divided by D and rearranged, then

$$\nabla^2\phi + \frac{\Sigma a}{D}(Ke^{-B_g^2 \tau} - 1)\phi = 0 \tag{7}$$

Equation (7) may be written in the form of the wave equation which the material buckling will satisfy.

$$\nabla^2\phi + B_m^2\phi = 0 \tag{8}$$

When the reactor is just critical, the two coefficients are equal, $B_g^2 = B_m^2$ [4, p. 47]. Then

$$B^2 = \frac{\Sigma_a}{D}\left(Ke^{-B^2\tau}-1\right) \tag{9}$$

By definition $\frac{D}{\Sigma_a}$ equals the diffusion length squared of neutrons in the reactor core (L^2), thus

$$B^2 = \frac{Ke^{-B^2\tau}-1}{L^2} \tag{10}$$

rearranging

$$1 + B^2L^2 = Ke^{-B^2\tau} \tag{11}$$

If B^2 is small, which would be the case if the reactor were large, then $e^{-B^2\tau}$ may be expanded in a series form

$$e^{-B^2\tau} = 1 - B^2\tau + \frac{(B^2\tau)^2}{2!} - \frac{(B^2\tau)^3}{3!} + \ldots \tag{12}$$

Neglecting all the terms in Eq. (12) beyond the second the following approximation is obtained

$$e^{-B^2\tau} \approx (1 + B^2\tau)^{-1} \tag{13}$$

Substituting Eq. (13) into Eq. (10) and rearranging yields

$$K = (1 + B^2L^2)(1 + B^2\tau) \tag{14}$$

Multiplying the right side of Eq. (14) and neglecting all terms higher than the second order

$$K = 1 + B^2L^2 + B^2\tau \tag{15}$$

rearranging

$$B^2 = \frac{K - 1}{L^2 + \tau} \tag{16}$$

Equation (16) gives the critical materials buckling for a

reactor where K is very close to unity, thus $K-1 \ll 1$. Imposing the condition $K-1 \ll 1$ normally makes Eq. (16) valid only for reactors fueled with natural or slightly enriched uranium. Reactors to which Eq. (16) applies are large in size.

IV. DEFINITIONS OF REACTOR DESIGN PARAMETERS

The methods of calculating the reactor constants required to solve Eq. (16) are presented in the following sections. It is interesting to note the interrelationships which exist between the various constants due to a dependence on one or more similar properties of the reactor. A knowledge of the properties of the system and their effects upon the reactor constants will be beneficial when the Buckingham Pi Terms are selected.

A. Neutrons Produced per Neutron Absorbed in Fuel [1, p. 83]

The average number of neutrons produced per neutron absorbed in the fuel is calculated by

$$\eta = \nu \frac{\Sigma_f}{\Sigma_{fuel}} \tag{17}$$

where ν = average number of fast neutrons released per slow neutron fission

Σ_f = macroscopic cross section for slow neutron fission

Σ_{fuel} = total macroscopic cross section for absorption of thermal neutrons in the fuel material.

B. Thermal Utilization [1, p. 84]

The thermal utilization (f) is defined by

$$f = \frac{\text{Thermal neutrons absorbed in fuel}}{\text{Total thermal neutrons absorbed}} \tag{18}$$

$$= \frac{\Sigma_{fuel}}{\sum_{i=1}^{n} \Sigma_{a_i}} = \frac{\Sigma_{fuel}}{\Sigma_a} \tag{19}$$

In this investigation η and f lose their individual identity since they are handled as the product. Combining Eq. (18) and (19)

$$\eta f = \nu \frac{\Sigma_f}{\Sigma_{fuel}} \cdot \frac{\Sigma_{fuel}}{\Sigma_a} \tag{20}$$

or canceling Σ_{fuel}

$$\eta f = \nu \frac{\Sigma_f}{\Sigma_a} \tag{21}$$

C. Resonance Escape Probability [1, p. 168]

The resonance escape probability (p) for uniform mixtures where both the fuel and moderator are exposed to the same neutron flux can be expressed as a function of the neutron energy by

$$p(E) = \exp\left[-\int_{E}^{E_0} \frac{\Sigma_{a_R}}{\xi(\Sigma_s + \Sigma_{a_R})} \frac{dE'}{E}\right] \tag{22}$$

where ξ = mean value of the average logarithmic energy decrement per collision (for the elements with mass larger than 10 the approximate equation $\xi = \frac{2}{A + 2/3}$ is valid; for lighter elements, $\xi =$

$1 + \left[\frac{(A-1)^2}{2A}\right]\left[\ln\left(\frac{A-1}{A+1}\right)\right]$ [1, p. 144])

A = mass of the scattering nuclei

Σ_s = macroscopic scattering cross section for neutrons during the slowing down process

Σ_{a_R} = macroscopic absorption cross section for neutrons during the slowing down process

E_o = average energy of fast neutrons produced by fission

E = energy where neutrons become thermalized (usually 0.025 ev)

For a system containing several nuclear species, $\bar{\xi}$ is defined by [1, p. 159]

$$\bar{\xi} = \frac{\sum_{i=1}^{n} \Sigma_{s_i} \xi_i}{\sum_{i=1}^{n} \Sigma_{s_i}} = \frac{\sum_{i=1}^{n} \Sigma_{s_i} \xi_i}{\Sigma_s} \tag{23}$$

The above form of the resonance escape probability is strictly applicable only to widely spaced, narrow resonances. For U^{238} and Th^{232}, Eq. (22) is a good approximation for the higher resonances, but breaks down at lower levels (approximately 10 to 100 ev) due to the concentration of resonance peaks. Normally p can be accurately evaluated by employing a semi-empirical equation to evaluate the integral in Eq. (22).

If the energy dependent macroscopic resonance absorption cross section is represented by $N_o \sigma_{a_o}$ and the macroscopic scattering cross section is assumed to be independent of energy, then it is possible to write

$$\frac{\Sigma_{a_R}}{\Sigma_s + \Sigma_{a_R}} = \frac{\Sigma_s}{\Sigma_s} \cdot \frac{N_o \sigma_{a_o}}{(\Sigma_s + \Sigma_{a_R})} \tag{24}$$

Substitution of Eq. (24) into Eq. (22) and removing N_o, Σ_s and $\bar{\xi}$--which are all independent of the neutron energy--outside the integral yields

$$p(E) = \exp\left[-\frac{N_o}{\bar{\xi}\,\Sigma_s} \int_E^{E_o} \frac{\sigma_{a_o}\,\Sigma_s}{\Sigma_s + \Sigma_{a_R}} \frac{dE}{E} \right] \tag{25}$$

The integral in Eq. (25) is known as the effective resonance integral (RI) and can be evaluated in terms of the scattering cross section per atom of resonance absorber for homogeneous reactors by the empirically determined relationships given below [5, p. 43] .

For U^{238}:

$$RI = 2.69 \left(\frac{\Sigma_s}{N_o}\right)^{0.471}, \quad 0 \leq \frac{\Sigma_s}{N_o} \leq 4000 \tag{26a}$$

$$\ln RI = 5.64 - \frac{163}{(\Sigma_s/N_o)^{0.65}}, \quad \frac{\Sigma_s}{N_o} > 4000 \tag{26b}$$

$$RI = 280, \qquad \frac{\Sigma_s}{N_o} = \infty \tag{26c}$$

For Th^{232}:

$$RI = 8.33 \left(\frac{\Sigma_s}{N_o}\right)^{0.253}, \quad 0 \leq \frac{\Sigma_s}{N_o} \leq 4500 \tag{27a}$$

$$RI = 70, \qquad 4500 \leq \frac{\Sigma_s}{N_o} = \infty \tag{27b}$$

where RI = effective resonance integral in barns

N_o = atoms per cubic centimeter of resonance ab-

sorber

$\frac{\Sigma_s}{N_o}$ = total scattering cross section, in barns, per atom of resonance absorber.

Normally the composition of homogeneous reactors is of such a nature that Eq. (26a) and (27a) provide sufficient range for calculating the effective resonance integral. The total resonance escape probability for uniformly dispersed systems is given by subetitution of Eq. (26a) and (27a) into Eq. (25). The simplified results are given below:

For U^{238}:

$$p = \exp\left[-\frac{2.69}{\xi} \left(\frac{N_o}{\Sigma_s} \right)^{0.529} \right] \tag{28a}$$

For Th^{232}:

$$p = \exp\left[-\frac{2.33}{\xi} \left(\frac{N_o}{\Sigma_s} \right)^{0.747} \right] \tag{28b}$$

If the exponents in Eqs. (28a) and (28b) are multiplied and divided by the thermal microscopic cross section for the resonance absorber (σ_{a_T}) taken to the proper power then the following results are obtained.

For U^{238}:

$$p = \exp\left[-\frac{2.69}{\xi \sigma_{a_T}} 0.529 \left(\frac{\sigma_{a_T} N_o}{\Sigma_s} \right)^{0.529} \right] \tag{29a}$$

For Th^{232}:

$$p = \exp\left[-\frac{2.33}{\xi \sigma_{a_T}} 0.747 \left(\frac{\sigma_{a_T} N_o}{\Sigma_s} \right)^{0.747} \right] \tag{29b}$$

Substitution of the thermal macroscopic absorption cross section of the resonance absorber ($\Sigma_{a_R}^{th}$) for $\sigma_{a_T} N_o$ and the values of σ_{a_T} for U^{238} and Th^{232}, which equal 2.75 and 7.0 barns respectively [3, p. 83], into Eqs. (29a) and (29b) gives the following results:

For U^{238}:

$$p = \exp\left[-\frac{1.57}{\bar{\xi}} \left(\frac{\Sigma_{a_R}^{th}}{\Sigma_s} \right)^{0.529} \right] \tag{30a}$$

For Th^{232}:

$$p = \exp\left[-\frac{1.94}{\bar{\xi}} \left(\frac{\Sigma_{a_R}^{th}}{\Sigma_s} \right)^{0.747} \right] \tag{30b}$$

D. Fast Fission Factor [1, p. 83]

By definition the fast fission factor (ϵ) equals the total number of fast neutrons produced by fissions due to neutrons of all energies divided by the number resulting from thermal-neutron fissions. For homogeneous reactors the fast fission factor has a value approximately equal to unity [6, p. 178].

Since the microscopic fission cross sections of U^{233}, U^{235} and Pu^{239} are small for fast neutrons (1.3 barns at fission energy increasing to approximately 9.0 barns at 1000 ev for U^{235}), the value of ϵ depends mainly upon the fission of the U^{238} and Th^{232} atoms by neutrons having energies above the fast fission threshold (approximately 1 Mev) [1, p. 43; 3, p. 127]. In the case of a uranium fueled reactor the

maximum value of ϵ occurs when the reactor composition is natural uranium. In this case $\epsilon = 1.19$ [1, p. 278; 3, p. 707] but, in most low enrichment heterogeneous reactors, ϵ varies from 1.00 to 1.05 depending on the size of the fuel rods [6, p. 195; 1, p. 279; 2, p. 95].

Because the variation of ϵ is slight, it will be assumed in this investigation that ϵ equals a constant. The error introduced in the infinite multiplication factor, K, by this assumption will be small providing the original value for ϵ is chosen judiciously.

E. Diffusion Length of Thermal Neutrons

By definition the diffusion length of thermal neutrons (L) equals [3, p. 201]

$$L = \frac{1}{\kappa} = \sqrt{\frac{D}{\Sigma_a}} \tag{31a}$$

where $\kappa = \sqrt{\frac{\Sigma_a}{D}}$ the relaxation length in the point-source solution of $\nabla^2\phi - \kappa^2\phi = 0$

D = the diffusion coefficient

Another useful definition involving L is the diffusion length squared which equals one-sixth of the mean square distance traveled by a neutron from the time that it first becomes thermalized until its capture [1, p. 116].

$$L^2 = \frac{1}{\kappa^2} = \frac{D}{\Sigma_a} \tag{31b}$$

For a weakly absorbing medium the total cross section is

large compared to the absorption cross section, then $\frac{\Sigma_a}{\Sigma} \ll 1$. In this case the diffusion coefficient is given by [3, p. 196]

$$D = \frac{1}{3\Sigma(1 - \bar{\mu})} \tag{32}$$

where $\bar{\mu}$ = the average cosine of the scattering angle per collision in the laboratory system.

$\Sigma = \Sigma_s + \Sigma_a \approx \Sigma_s$

If the medium contains several nuclei, each possessing a distinct Σ and $\bar{\mu}$ value, then the diffusion coefficient can be determined by

$$\frac{1}{D} = 3\sum_{i=1}^{n} \Sigma_i(1 - \bar{\mu}_i) \tag{33}$$

F. Neutron Age [1, p. 181; 7, p. 4]

The age (τ) equals one-sixth of the mean squared distance traveled by a neutron from the point of its origin at fission energy ($\tau = 0$) to the point where its age is τ .

The value of the age for a specific material can be determined experimentally from the equation for the slowing down density (q) of a point source of neutrons in an infinite medium [1, p. 180].

$$q(r,\tau) = \frac{S e^{-\frac{r^2}{4\tau}}}{(4\pi\tau)^{3/2}} \tag{34}$$

The slowing down density is defined as the number of neutrons per unit volume per unit time that slow down past a given energy [1, p. 150]. Relationships between the neutron flux

and the slowing down density for the various cases of interest can be derived [1, pp. 151-65], thus measurements of the neutron flux with cadmium covered indium foils provide the necessary data to evaluate q. For neutrons of a specified age, Eq. (34) reduces to

$$\ln q(r) = \text{const.} - \frac{r^2}{4\tau} \qquad (35)$$

A plot of indium foil activity vs r^2 yields a straight line on semi-logarithmic graph paper having a slope equal to $-\frac{1}{4\tau}$.

Unfortunately, the number of experimentally determined ages for numerous material mixtures and various neutron energies is extremely limited. Consequently, a number of methods have been developed to predict theoretically the age of neutrons in multi-material systems [2, pp. 268-96].

It is extremely difficult to develop theoretical equations which can be solved to predict accurately τ for mixtures of hydrogen and moderately heavy elements such as oxygen. The age theory can be used to predict quite accurately the age of neutrons in moderators composed of moderately heavy nuclei such as graphite, but a decrease in accuracy is noted as the moderator nuclei become lighter. Erroneous results are obtained when the age theory is applied to moderators containing even slight amounts of hydrogen, since the neutron energy reduction process can no longer be approximated by a continuous curve.

Several theoretical equations for calculating the age have been developed based on approximations of the general transport equation, but these equations are all extremely complicated and are not easily evaluated. Such methods as the diffusion approximation of the transport equation, the B_3 approximation, the Greuling-Goertzel and the Goertzel-Selengut methods fit into this category. Considerable calculative brevity and a high degree of accuracy can be achieved by using a semi-emperical method to calculate the age.

In this investigation the Flügge-Tittle method for estimating the neutron age in mixtures of heavy elements with hydrogen is utilized [2, p. 278]. The Flügge-Tittle method is based on the assumption that the slowing down of the fast neutrons during the first few collisions depends upon the energy dependent scattering cross section of hydrogen and the average energy loss per collision with a hydrogen atom. Below a neutron energy level of 100 kev the age theory yields an accurate value for the age, even in hydrogenous substances. The total age is the summation of the slowing down lengths over adjoining energy increments down to approximately 100 kev plus the age calculated by the age equation from 100 kev down to thermal energy.

For a point source the number of neutrons which have not collided at a given distance r from the source can be expressed in terms of the flux by

$$\phi = \frac{Se^{-\frac{r}{\lambda_1}}}{4\pi r^2} \tag{36}$$

where λ_1 equals the mean free path for the neutrons in a specified energy increment. Since the collision rate per unit volume at distance r is given by $\Sigma_s\phi$, the second spatial moment of the collision rate is given by

$$r_1^2 = \frac{\int_0^\infty \phi\Sigma_s r^2 dV}{\int_0^\infty \phi\Sigma_s dV} \tag{37}$$

where r_1^2 equals the mean square displacement for first collisions. By substituting Eq. (36) into Eq. (37) and carrying out the indicated integration

$$r_1^2 = 2\lambda_1^2 \tag{38}$$

The average energy of the neutron after each collision with a hydrogen atom is calculated from the definition of the average logarithmic energy decrement for a pure material [1, p. 143]

$$\xi = \ln \frac{E_1}{E_2} \tag{39}$$

where E_1 = energy of neutron before collision

E_2 = energy of neutron after collision

Since ξ for hydrogen equals one, $\frac{E_1}{E_2}$ equals 2.71. Thus the mean square displacement required to reduce the average neu-

tron energy by a factor of 2.71 equals $2\lambda_1^2$. If several collisions are necessary to reduce the neutron energy to approximately 100 kev, the total mean square displacement is given by summing the individual mean square displacements for each adjoining energy increment. The slowing-down length is given by

$$l_s = \frac{\sum_{i=1}^{n} r_i^2}{6} = \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots}{3} \qquad (40)$$

The various λ's for a mixture containing hydrogen and additional substances can be evaluated by the empirical equation

$$\frac{\lambda_i}{\lambda} = 2.155 - 0.119 \frac{\lambda_o}{\lambda_h} \qquad (41)$$

where λ_i = average mean free path for the neutrons

λ = mean free path total for all constituents

λ_o = mean free path for all nonhydrogenous components

λ_h = mean free path for all hydrogenous components

The total slowing-down length to any arbitrary neutron energy level below 100 kev is given by

$$L_s^2 = l_s^2 + \tau(100 \text{ kev} - E) \qquad (42)$$

The age in a mixture of nuclei is given by (see appendix for derivation of the age equation)

$$\tau = \int_{E}^{100\ \text{kev}} \frac{\frac{dE}{E}}{3(\overline{\Sigma_S \xi})(\overline{\Sigma_S T})} \tag{43}$$

where $T = 1 - \bar{u}$

$$\overline{\Sigma_S \xi} = \sum_{i=1}^{n} \Sigma_{S_i} \xi_i$$

$$\overline{\Sigma_S T} = \sum_{i=1}^{n} \Sigma_{S_i} T_i$$

The Flügge-Tittle method gives excellent agreement with experimental data provided that 40 to 45 percent of the moderator atoms are hydrogen. When the hydrogen atom content is less than the above stated value, the scattering ability of the nonhydrogenous materials becomes appreciable. The scattering effect of the nonhydrogenous materials is partially considered in the calculated value of the various λ's used in Eq. (41), but the average energy loss of the neutron per collision is still considered to be equal to the value for hydrogen. Obviously, the decreased accuracy of the Flügge-Tittle method of age estimation with reduced hydrogen concentration results from the incorrect evaluation of the average energy loss per collision.

In this investigation the Flügge-Tittle method was modified to account for the reduced average energy loss per collision when the hydrogen atom concentration was less than 41.7 percent (this corresponds to 50 percent H_2O by volume in H_2O-

Al mixture). The following procedure was used to calculate an artificial value for the average energy loss per collision. An average value of $\bar{\xi}$, calculated by Eq. (23), for the nonhydrogenous constituents was determined for the energy range being investigated. The value of $\frac{E_1}{E_2}$ for this average $\bar{\xi}$ was resolved. The mid-point between the $\frac{E_1}{E_2}$ values for the nonhydrogenous constituents and hydrogen was determined. Then a straight line interpolation method was employed from the hydrogen $\frac{E_1}{E_2}$ value to the mid-point $\frac{E_1}{E_2}$ value; the former was used for a 50 percent by volume water mixture (41.7 percent hydrogen atom content), the latter for mixtures containing no hydrogenous materials. As the percent of hydrogen contained in the system decreased below 41.7 percent, the value of $\frac{E_1}{E_2}$ also decreased.

Comparative values of the age calculated by the Flügge-Tittle and the Modified Flügge-Tittle methods are compared with experimentally determined values of the age in Table 1. Excellent correlation of the Flügge-Tittle method between 50 and 100 percent water and the Modified Flügge-Tittle method when the water percentage falls below 50 percent was noted for various materials except in the case where D_2O was used to dilute the water.

Table 1. Comparison of experimental and theoretically determined values for the neutron age

Energy Range	% Metal by Volume	Metal	$\%H_2O$ by Volume	τ by Flügge-Tittle Method	τ by Modified Flügge-Tittle Method	τ by Experiment
2 Mev	0	Al	100	29.36[a]	--	30.8[b]
to	20	Al	80	40.74[a]	--	
1.44 ev	33	Al	67	52.50[a]	--	49.6[b]
	50	Al	50	73.70[a]	--	76.8[b]
4 Mev	50	Al	50	99.1	99.1	100.2[c]
to 1.44	67	Al	33	128.0	144.9	143.5[c]
ev	75	Al	25	172.4	192.0	197.3[c]
	75	Fe	25	78.2	86.7	83.0[c]
	48.6	D_2O	51.4	79.1	--	72.5[d]
2 Mev	48.6	D_2O	51.4	43.2	--	38.6[d]
to 1.44	100	C	0	--	336.3	311.0[e]
ev	100	Be	0	--	102.2	97.0[f]

[a](8, p. 25).

[b](9, p. 12).

[c](10, p. 5).

[d](11, p. 21).

[e](3, p. 331).

[f](7, p. 9).

V. APPLICATION OF DIMENSIONAL ANALYSIS TO SINGLE GROUP REACTOR THEORY

In this section dimensional analysis will be applied to the modified single group critical reactor equation, Eq. (54). A method will be developed which utilizes the resulting dimensionless quantities to design a model reactor which will retain the same dynamic operating characteristics as the prototype. Each of the dimensionless quantities or Pi terms will be examined to discover the method and effect of varying one of its factors. Also the interrelationship and allowable distortion of each Pi term will be determined. Then the maximum size variation of a model reactor will be investigated. Finally, a sample model problem will be presented to augment the theoretical discussion.

A. Review of Dimensional Analysis

A functional relationship between the dependent quantity α and the various independent quantities which affect the magnitude of α may be written as

$$\alpha = f(C_1, C_2, C_3, \ldots, C_n) \qquad (44)$$

where the independent quantities are represented by the various C values. If the functional relationships of the various independent quantities are known and the evaluation of these relationships are not prohibitive, then the value of α can be determined. However, many functional relationships are not easily evaluated so that the calculation of α by Eq. (44) may

be very costly and time consuming, or sometimes impossible.

The Buckingham Pi Theorem offers a method of grouping the various related quantities into dimensionless groups, thus reducing the number of variables which must be examined. Basically, ". . . the Buckingham Pi Theorem states that the number of dimensionless and independent quantities required to express a relationship among the variables in any phenomena is equal to the number of quantities involved, minus the number of dimensions in which those quantities may be measured" [12, p. 36]. Algebraically the Pi Theorem is expressed

$$s = n - b \tag{45}$$

where s = number of required dimensionless terms called Pi terms

n = total number of quantities involved

b = number of basic dimensions involved

The form of Eq. (44) becomes

$$\pi_1 = f(\pi_2, \pi_3, \pi_4, \ldots, \pi_s) \tag{46}$$

If it is assumed that Eq. (46) gives the relationships for the prototype, and that the variables and their functional relationships are maintained, then a similar equation will also apply to a model of the prototype [12, p. 58]. The equation for the model is

$$\pi_{1_m} = f(\pi_{2_m}, \pi_{3_m}, \pi_{4_m}, \ldots, \pi_{s_m}) \tag{47}$$

If the model is designed and operated so that

$$\pi_{2_m} = \pi_2 \tag{48}$$

$$\pi_{3_m} = \pi_3$$

$$\pi_{4_m} = \pi_4$$

$$\cdots = \cdots$$

$$\pi_{s_m} = \pi_s$$

then

$$f(\pi_{2_m}, \pi_{3_m}, \pi_{4_m}, \ldots, \pi_{s_m}) = f(\pi_2, \pi_3, \pi_4, \ldots, \pi_s) \tag{49}$$

and by substitution of Eqs. (46) and (47) into Eq. (49)

$$\pi_{1_m} = \pi_1 \tag{50}$$

Equation (50), known as the prediction equation, will be advantageously utilized in the following sections of this investigation to calculate the radius of a critical reactor model.

B. Determination of Reactor Pi Terms

For a spherical reactor the buckling equals [1, p. 214]

$$B^2 = \left(\frac{\pi}{R}\right)^2 \tag{51}$$

The over-all multiplication factor (K) equals

$$K = \epsilon \eta f p \tag{52}$$

Substitution of the quantities previously derived for ηf and p in Eqs. (21) and (30a) or (30b) yields

$$K = \epsilon(\nu \frac{\Sigma_f}{\Sigma_a}) \left\{ \exp \left[- \frac{\text{const.}}{\bar{\xi}} \left(\frac{\Sigma_{a_R}^{th}}{\Sigma_s} \right)^{\text{const.}} \right] \right\} \tag{53}$$

The radius of a critical reactor can be determined by substituting Eqs. (31b), (51), and (53) into Eq. (16)

$$\left(\frac{\pi}{R}\right)^2 = \frac{\epsilon(\nu \frac{\Sigma_f}{\Sigma_a}) \left\{\exp\left[-\frac{\text{const.}}{\bar{\xi}}\left(\frac{\Sigma_{a_R}^{th}}{\Sigma_a}\right)^{\text{const.}}\right]\right\} - 1}{\frac{D}{\Sigma_a} + \tau} \qquad (54)$$

Seven or eight dimensionless reactor design terms could be formed by multiplying both sides of Eq. (54) by the radius squared and rearranging. This method of designing model reactors was not undertaken because more versatile Pi terms could be obtained by employing dimensional analysis.

From Eq. (54) the radius of a critical reactor can be written in terms of a functional relationship similar to Eq. (44)

$$R = f(\epsilon, \nu, \Sigma_f, \Sigma_a, \bar{\xi}, \Sigma_{a_R}^{th}, \Sigma_a, D, \tau) \qquad (55)$$

The dimensions for the various quantities given in Eq. (55) are listed in Table 2.

Table 2. Dimensions of quantities used to design nuclear reactors utilizing single group theory

Quantities	Dimensions
R, reactor radius	Length
ϵ, fast fission factor	--
ν, average number of fast neutrons released per neutron fission	--
Σ_f, macroscopic fission cross section	Length^{-1}
Σ_a, macroscopic absorption cross section	Length^{-1}
ξ, logarithmic energy decrement	--
$\Sigma_{a_R}^{th}$, macroscopic resonance cross section	Length^{-1}
Σ_s, macroscopic scattering cross section	Length^{-1}
D, diffusion coefficient	Length
τ, neutron age	Length^{+2}

By application of Eq. (45) the ten quantities in Eq. (55) require nine independent Pi terms to describe the system completely.

The nine Pi terms, determined by inspection of Eq. (54), are listed in Table 3.

Table 3. Design π_i terms employed in single group reactor analysis

π_1	$R\Sigma_a$
π_2	ϵ
π_3	ν
π_4	ξ
π_5	$\frac{\Sigma_f}{\Sigma_a}$
π_6	$\frac{\Sigma_a^{th}}{\Sigma_a}$
π_7	$\frac{\Sigma_f}{\Sigma_{a_R}^{th}}$
π_8	$\Sigma_a D$
π_9	$\frac{1}{D^2}$

If the above Pi terms are used to determine the critical radius of a model reactor which duplicates the operating characteristics of the prototype, then by Eq. (46)

$$R\Sigma_a = f(\epsilon, \nu, \xi, \frac{\Sigma_f}{\Sigma_a}, \frac{\Sigma_a^{th}}{\Sigma_a}, \frac{\Sigma_f}{\Sigma_{a_R}^{th}}, \Sigma_a D, \frac{1}{D^2}) \qquad (56)$$

If the model and prototype are operated so that Eq. (48) is valid, then

$$\epsilon_m = \epsilon$$

$$\nu_m = \nu \tag{57}$$

$$\xi_m = \xi$$

$$\frac{\Sigma_{f_m}}{\Sigma_{a_m}} = \frac{\Sigma_f}{\Sigma_a}$$

$$\frac{\Sigma_{a_m}^{th}}{\Sigma_{a_m}} = \frac{\Sigma_{a_R}^{th}}{\Sigma_a}$$

$$\frac{\Sigma_{f_m}}{\Sigma_{a_m}^{th}} = \frac{\Sigma_f}{\Sigma_{a_R}^{th}}$$

$$\Sigma_{a_m} D_m = \Sigma_a D$$

$$\frac{\tau_m}{D_m^2} = \frac{\tau}{D^2}$$

Then by Eq. (50)

$$R_m \Sigma_{a_m} = R \Sigma_a \tag{58a}$$

or rearranging

$$R_m = R \frac{\Sigma_a}{\Sigma_{a_m}} \tag{58b}$$

If both model and prototype are constructed of identical materials and constituent ratios and operated at the same temperatures, then all the Pi terms in Eq. (57) are equal. Also $\Sigma_a = \Sigma_{a_m}$; therefore $R_m = R$. Consequently, no reduction in size of the model is achieved unless the material parameters of the model are modified from those of the prototype.

Variation of the model material parameters is achieved by modification of one or more of the following items: the oper-

ating temperature, the ratio of the reactor materials or the type of reactor materials. Of the three suggested methods, only the latter two are suitable for drastically altering the size of the model.

C. Discussion of Pi Terms

By Eq. (58b) any variation of the $\frac{\Sigma_a}{\Sigma_{a_m}}$ ratio directly affects the size of the model relative to the prototype. To decrease the model size the ratio $\frac{\Sigma_a}{\Sigma_{a_m}}$ must be decreased, conversely an increase in $\frac{\Sigma_a}{\Sigma_{a_m}}$ increases R_m. Thus a method of varying the ratio $\frac{\Sigma_a}{\Sigma_{a_m}}$ without affecting the equality of the other eight Pi terms in Eq. (57) is required if the model size is to be varied from that of the prototype.

1. ϵ and ν

Selection of the proper value for ϵ has been discussed previously.

The value of ν is independent of the fuel enrichment but dependent upon the type of fuel. Values of ν for the various thermal reactor fuels are given in Table 4 [3, p. 124].

Table 4. Thermal neutron values of ν (2200m/sec)

Fuel	ν
U^{233}	2.51
U^{235}	2.47
U^{239}	2.91

If the model contains the same fuel as the prototype, then the equality of the v Pi term is maintained.

2. $\bar{\xi}$

The value of $\bar{\xi}$, calculated by Eq. (23), is dependent upon the macroscopic cross section and the concentration of the nuclei present in the reactor. If the volume percent of materials is different in the model and the prototype, then the respective values of $\bar{\xi}$ will be affected.

Since $\bar{\xi}$ appears only in the exponential of Eq. (54), a five to ten percent distortion of the $\bar{\xi}$ term will not alter the radius more than one percent. The exact allowable distortion of $\bar{\xi}$ is dependent upon the value of the exponential term; greater distortion is possible when the exponential factor equals 0.9 than when it equals 0.6.

A distortion of $\bar{\xi}$ will not vary directly the $\frac{\Sigma_a}{\Sigma_{a_m}}$ ratio, but it does allow some latitude in the value of Σ_g. The importance of the limited freedom of Σ_g will be discussed in a following section.

3. $\frac{\Sigma_f}{\Sigma_a}$

This Pi term is relatively easy to vary with a high degree of accuracy. The value of Σ_f may be altered by either changing the volume percentage of fuel or by varying the fuel enrichment. It will be shown in the following section that only the variation of the volume percentage of fuel is valid

for an investigation where the model and prototype are operated at the same temperature. If the operating temperature of the model is different from that of the prototype, the variation of Σ_f is usually accomplished by altering both the volume percent and enrichment of the fuel.

The value of Σ_a depends on the type and quantity of nuclei present. For a system composed of a number of nuclei Σ_a can be evaluated by the volume-fraction weighted average method.

If the case arises where all the Pi terms except $\frac{\Sigma_f}{\Sigma_a}$ are equalized and the inequality of the $\frac{\Sigma_f}{\Sigma_a}$ Pi term is caused by the low value of Σ_a, then corrective measures may be taken by the addition of the proper amount of an absorber (for example, boron) to the system. The high ratio of $\frac{\Sigma_a}{\Sigma_a}$ for boron allows a significant change in the system Σ_a value per minute variation of Σ_a.

4. $\frac{\Sigma_f}{\Sigma_{a_R}^{th}}$

This Pi term is dependent upon fuel enrichment and operating temperature. If both the model and prototype are operated at the same temperature, then the cross section properties of the fuel in both reactors are identical. In this case the equality of this Pi term is preserved only if both model and prototype contain fuel of the same enrichment. Since U^{235} is a $\frac{1}{v}$ absorber, while U^{238} is a non-$\frac{1}{v}$ absorber, any deviation

of the model operating temperature from that of the prototype will affect the value of Σ_s. A lower model operating temperature will increase Σ_s but will not appreciably affect $\Sigma_{a_R}^{th}$.

5. $\frac{\Sigma_{a_R}^{th}}{\Sigma_s}$

This is another Pi term which appears in the exponential of Eq. (54); it also can be distorted without causing a significant variation in the reactor radius. The value of $\Sigma_{a_R}^{th}$ is fixed by the fuel enrichment and the operating temperature, but Σ_s may vary. Actually, a 10 to 15 percent variation in Σ_s changes the calculated radius by less than one percent. A variation in Σ_s which would compensate for the distortion of $\bar{\xi}$ would increase the accuracy of the reactor radius computed by the dimensional analysis method.

6. $\Sigma_s D$

The value of Σ_s was discussed in the preceding section; however, any distortion of its value in the $\frac{\Sigma_{a_R}^{th}}{\Sigma_s}$ Pi term will also effect the value of D in this Pi term.

With reference to Eq. (32), D is dependent upon Σ_s and $(1-\bar{u})$. Values of $(1-\bar{u})$ vary from approximately 0.3 for hydrogen to 1.0 for the heavy elements while the useful reactor materials possess values of Σ_s which vary from 3.45 cm^{-1} for water to 0.08 cm^{-1} for aluminum. Since coupling between Σ_s and D exists in Eq. (32), any material changes which vary Σ_s also affect the value of D. Considering the variation of

(1-6) and Σ_s values, it is difficult to offset an increase in Σ_s with an equal decrease of D. For example, the values of Σ_s and D given in Table 5 are computed for a 100 percent water system and a homogeneous system composed of 43.7 percent water and 56.3 percent graphite by volume [13].

Table 5. Comparison of values of Σ_s and D for H_2O and H_2O-C systems

System Composition	Σ_s	D
100% H_2O	3.45	0.143
43.7% H_2O + 56.3% C	1.725	0.272

In Table 5 Σ_s decreased by 50 percent while D increased 190 percent. The value of D in the water-graphite system is 4.9 percent too small to maintain the equality of the $\Sigma_s D$ Pi term in the two systems. A third material must be added to the water-graphite system to adjust the value of D if the Pi term equality is to be maintained.

7. $\frac{\tau}{D^2}$

This Pi term limits the size changes which can be achieved by the model. If D changes by a factor of two, equality of the Pi term requires that τ change by a factor of four. If the major moderating material of the model is to be water (fission to thermal age 31.4 cm^2), then a reduction of D by a factor of two in the model requires that the prototype have a minimum fission to thermal age of 126 cm^2. It can be clearly

shown by an examination of the thermal-nuclear properties of the elements[13] that it would be impossible to achieve the above proposed model size change unless the moderator material in the prototype contained a reasonable percentage of graphite. Large reductions in the size of the model require that the prototype moderator be composed almost exclusively of graphite. Table 6 lists the possible reactor moderating materials with the respective ages of each [2, p. 123; 3, p. 331].

Table 6. Fission to thermal age for neutrons

Moderator	τ (cm^2)
H_2O	31.4
D_2O	125
Be	97.2
BeO	105
C	364

The distortion of Σ_s affects the value of τ as the square of the decimal variation of D minus 1.00. If Σ_s is 10 percent too small, then D is 10 percent too high, and τ is (1.21 - 1.00) = 0.21 or 21 percent too high. Similarly a Σ_s value 10 percent too high causes a 19 percent decrease in τ. For the Pi terms shown in Table 3, a distortion of Σ_s will not affect the $\frac{\Sigma_a}{\Sigma_{s_m}}$ ratio of Eq. (38b); thus, the model radius will not be affected.

In some cases the distortion of Σ_s and the resulting ef-

fect on τ may be beneficial if π_1 shown on Table 3 is altered to $R\Sigma_s$. The model radius in Eq. (58b) would be given by

$$R_m = R \frac{\Sigma_s}{\Sigma_{s_m}} \tag{59}$$

If Σ_{s_m} were distorted 10 percent as explained above, the value of R_m would be changed a similar amount.

D. Model Limitations

Since the size of a reactor is dependent upon the nuclear properties of the construction materials, the maximum size variation which can be achieved by a model is dictated by the prototype materials. Generally, a model can be made smaller if the prototype moderating material is Be, BeO or C, or a combination of these materials. Conversely, an enlarged model is used when the prototype moderating material is H_2O or D_2O. No specific rule governs the model size when the prototype moderating material is a combination of H_2O or D_2O plus Be, BeO or C.

In general, a reduction in the model size becomes progressively harder for decreasing values of the prototype's τ. The converse applies to increasing model sizes.

It has been determined that for decreasing model sizes a model to prototype radius ratio $(\frac{R_m}{R})$ of 0.5 is the minimum that can be obtained for thermal reactors. Under normal design conditions, a radius ratio from 0.57 to 0.67 can be achieved. Since the volume of a spherical reactor is propor-

tional to the radius cubed, the radius ratio can be used to compute the size of the model compared to the prototype by

$$\frac{V_m}{V} = \left(\frac{R_m}{R}\right)^3 \tag{60a}$$

Substitution of Eq. (58b) and rearranging gives

$$\frac{V_m}{V} = \left(\frac{\Sigma_a}{\Sigma_{a_m}}\right)^3 \tag{60b}$$

The volume ratio for the normal radius ratios given above are 0.185 and 0.30 respectively. For increasing model sizes the volume ratio is the reciprocal of the above values (5.4 and 3.3 respectively).

Similar increases or decreases in the model size can be achieved with cylindrical and rectangular reactor shapes by the substitution of the proper expression for the buckling in Eq. 54 [1, p. 214].

$$B^2_{cyl.} = \left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2 \tag{61}$$

$$B^2_{rect.} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2$$

where H = height of cylinder

a,b,c = length, width and height of rectangle.

If the buckling expressions given in Eq. (61) are substituted into Eq. (16), then the additional quantities H, a, b and c (all have dimensions of length) appear in Table 2. To satisfy the Buckingham Pi Theorem the following additional Pi terms would appear in Table 3.

For a cylinder:

add $$\pi_{10} = H\Sigma_a \tag{62a}$$

For a rectangle:

change $$\pi_9 = a\Sigma_a \tag{62b}$$

$$\pi_{10} = b\Sigma_a$$

add

$$\pi_{11} = c\Sigma_a$$

By arranging the Pi terms listed in Eqs. (62a) and (62b) in the form of Eq. (58b), then

$$H_m = H \frac{\Sigma_a}{\Sigma_{a_m}} \tag{63a}$$

$$a_m = a \frac{\Sigma_a}{\Sigma_{a_m}} \tag{63b}$$

$$b_m = b \frac{\Sigma_a}{\Sigma_{a_m}}$$

$$c_m = c \frac{\Sigma_a}{\Sigma_{a_m}}$$

Note that the dimensional relationships $\left(\frac{H_m}{H}, \frac{a_m}{a}, \frac{b_m}{b}, \frac{c_m}{c}\right)$ all equal $\frac{\Sigma_a}{\Sigma_{a_m}}$. The volume ratios for cylindrical and rectangular reactors are:

$$\left(\frac{V_m}{V}\right)_{cyl.} = \left(\frac{R_m}{R}\right)^2 \left(\frac{H_m}{H}\right) \tag{64a}$$

$$\left(\frac{V_m}{V}\right)_{rect.} = \left(\frac{a_m}{a}\right)\left(\frac{b_m}{b}\right)\left(\frac{c_m}{c}\right) \tag{64b}$$

By substitution of the relationships given by Eqs. (63a) and

(63b) into Eqs. (64a) and (64b) respectively and rearranging, the volume ratio for both cylindrical and rectangular reactors is given by

$$\frac{V_m}{V} = \left(\frac{\Sigma_a}{\Sigma_{a_m}}\right)^3 \tag{65}$$

Comparing Eq. (65) with Eq. (60b) it is seen that the volume ratio for the three standard reactor shapes equals the cube of the $\frac{\Sigma_a}{\Sigma_{a_m}}$ ratio.

It was previously stated that it is impossible to vary independently the fuel enrichment of the prototype and model without varying the operating temperature of the model from that of the prototype. The fuel enrichment may be altered if the $\frac{\Sigma_f}{\Sigma_{a_R}^{th}}$ term is eliminated and the $\frac{\Sigma_{a_R}^{th}}{\Sigma_s}$ is modified. Assuming that the absorption cross section of the moderator material is negligible compared to that of the fuel, then

$$\Sigma_{a_R}^{th} = \Sigma_a - \Sigma_f \tag{66}$$

Substitution of Eq. (66) into Eqs. (30a) and (30b) changes the form of Eq. (54) to

$$\left(\frac{\pi}{R}\right)^2 = \frac{\epsilon\left(\nu \frac{\Sigma_f}{\Sigma_a}\right)\left\{\exp\left[-\frac{\text{const.}}{\bar{\xi}}\left(\frac{\Sigma_a}{\Sigma_s} - \frac{\Sigma_f}{\Sigma_s}\right)^{\text{const.}}\right]\right\} - 1}{\frac{D}{\Sigma_a} + \tau} \tag{67}$$

The resulting changes in the Pi terms in Table 3 are

$$\pi_6 = \frac{\Sigma_{a_R}^{th}}{\Sigma_g} \text{ modified to equal } \frac{\Sigma_R}{\Sigma_g} \qquad (68)$$

$$\pi_7 = \frac{\Sigma_f}{\Sigma_{a_R}^{th}} \text{ eliminated}$$

It should be noted that the $\frac{\Sigma_f}{\Sigma_g}$ term appearing in the exponential of Eq. (67) can be evaluated by multiplying π_5 ($\frac{\Sigma_f}{\Sigma_a}$, Table 3) by the modified π_6 ($\frac{\Sigma_R}{\Sigma_g}$, Eq. (68)).

With the modified Pi terms given in Eq. (68), it is possible to alter the fuel enrichment of the model and prototype and maintain the equality of the remaining Pi terms.

E. Example Reactor Problem

An example reactor problem has been included to provide some insight into the degree of prediction accuracy which can be achieved using this model analysis method.

The initial operating conditions for a one-region molten-salt prototype reactor are given below [5, pp. 567-592]:

Prototype reactor composition by weight

NaF	= 19.54%
ZrF_4	= 74.6%
UF_4	= 5.86%
Initial enrichment $\left(\frac{U^{235}}{U_{total}}\right)$	= 0.7
Poison fraction $\left(\frac{\Sigma_a \text{ (poisons)}}{\Sigma_f \text{ (fuel)}}\right)$	= 0.07
Reactor operating temperature	= 1150°F

Reactor radius	= 414 cm
Approximate thermal power with maximum flux of 10^{12}	= 255 mw

The various macroscopic cross sections were evaluated by the volume-fraction weighted average method using temperature corrected thermal microscopic cross section data [2, p. 177; 13]. The value of ϵ equaled approximately 1.00 for a homogeneous reactor, ν was evaluated from Table 4, and $\bar{\xi}$, D and τ were calculated by the equations previously presented. It should be noted that the microscopic cross section data used to calculate the λ values in the τ determination for the high energy neutrons was taken from the report BNL-325 [14].

The properties of the reactor materials are listed in Table 7 for the prototype reactor and the ideal and actual model reactors.

The properties of the model reactor material for $\frac{R}{R_m} =$ 1.75 were calculated from the following information:

Model reactor composition by weight

C	= 13.5%
Zr	= 53.2%
U	= 33.3%
Initial enrichment $\left(\frac{U^{235}}{U_{(total)}}\right)$	= 0.574
Reactor operating temperature	= 77°F
Reactor radius	= 226 cm

Table 7. Reactor constants for prototype and model

Constants	Prototype[b]	Ideal Model Constants for $\frac{R}{R_m} = 1.75$	Actual Model[a] Constants for $\frac{R}{R_m} = 1.75$
ϵ	1.00	1.00	1.00
ν	2.47	2.47	2.47
Σ_g	0.0877cm^{-1}	0.1316cm^{-1}	0.1316cm^{-1}
$\Sigma_{a_R}^{th}$	0.000314cm^{-1}	0.000471cm^{-1}	0.000517cm^{-1}
Σ_a	0.1167cm^{-1}	0.175cm^{-1}	0.175cm^{-1}
Σ_s	0.2655cm^{-1}	0.3982cm^{-1}	0.3617cm^{-1}
$\tilde{\xi}$	0.079	0.079	0.094
D	1.29cm	0.861cm	0.953cm
τ	1016cm^2	452cm^2	509cm^2
R	414cm	236cm	226cm

[a] 13.5% C + 53.2% Zr + 33.3% U by weight.

[b] 19.54% NaF + 74.6% ZrF_4 + 5.86% UF_4 by weight.

The properties of the ideal model for $\frac{R}{R_m} = 1.75$ were calculated from the properties of the prototype materials. Several combinations of materials were tried, varying both material and composition, to obtain the actual constants which would fit best the ideal model conditions. The majority of actual model constants agreed with the ideal design values except for $\Sigma_{a_R}^{th}$, $\bar{\xi}$, D and τ. Even these constants were erroneous by only 10 to 13 percent except for $\bar{\xi}$ which was approximately 20 percent in error.

Of the four constants only $\Sigma_{a_R}^{th}$ was easily distorted without affecting the value of any of the other reactor material constants. The matching of the $\Sigma_{a_R}^{th}$ constant could be achieved by allowing the radius to distort. Note that it was reasonable to distort $\Sigma_{a_R}^{th}$ as shown because of the distortion of $\bar{\xi}$. Since the model $\bar{\xi}$ value was larger than it ideally should be, this means that the neutrons are not in the resonance energy region as long as they would be in the prototype; thus the resonance escape probability would be greater in the model. By adding more U^{238} atoms, the model resonance escape probability was reduced to a value which was almost equal to the value existing in the prototype.

VI. EXTENSION OF DIMENSIONAL ANALYSIS TO MULTI-GROUP REACTOR THEORY

A. Formulation of the Multi-Group Critical Equation

If the fission to thermal energy range is divided into several energy increments, the neutrons within each increment at any specific instant will obey approximately the steady state diffusion equation [2, p. 110].

$$D\nabla^2\phi - \Sigma\phi + S = 0 \tag{69}$$

In the above equation it was assumed that the fission neutrons all have the same energy, the fission neutrons all enter the highest energy group, and no absorption of neutrons takes place in any energy group except the slowest or thermal group.

A fission energy neutron is removed from the fast energy group either by leaking out of the reactor or by decreasing its energy through collisions with the moderator atoms. The former is considered by the $D\nabla^2\phi$ term in Eq. (69), the latter by the $\Sigma\phi$ term. When a neutron is removed from one group, it is assumed that it enters the next lower energy group. It is evident that a loss of neutrons in the fission energy group becomes a neutron source for the next lower energy increment. This neutron cascading process continues down to the thermal energy group. Here the neutrons given up by the preceding energy group are adsorbed by the fuel and cause fission, producing fast neutrons. Suppose there are n energy groups, such

that 1, 2, ..., i, ..., n, where 1 is the fission energy group and n represents the thermal group. Then the general diffusion equation is given by

$$D_i\nabla^2\phi_i - \Sigma_i\phi_i + \Sigma_{i-1}\phi_{i-1} = 0 \qquad (70)$$

where Σ_i = the slowing down cross section in all cases except for i = n

Σ_n = the true absorption cross section for thermal neutrons

For the fission energy group, i = 1, Eq. (70) is not valid, because an average of more than one neutron is produced per atom fissioned. Therefore, the fission energy source term must be multiplied by K. The fission energy diffusion equation is given by

$$D_1\nabla^2\phi_1 - \Sigma_1\phi_1 + K\Sigma_n\phi_n = 0 \qquad (71)$$

The neutron flux for each group satisfies the wave equation

$$D_i\nabla^2\phi_i + B^2\phi_i = 0 \qquad (72)$$

where the buckling constants are equal for all the groups.

Substitution of Eq. (72) into Eqs. (70) and (71) gives

$$\begin{array}{ccccccccc}
-(D_1B^2+\Sigma_1)\phi_1 & + & 0 & + & 0 & + & K\Sigma_n\phi_n & = & 0 \\
\Sigma_1\phi_1 & - & (D_1B^2+\Sigma_1)\phi_1 & + & 0 & + & 0 & = & 0 \\
0 & + & \Sigma_1\phi_1 & - & \cdots & + & 0 & = & 0 \\
0 & + & 0 & + & \cdots & - & (D_nB^2+\Sigma_n)\phi_n & = & 0
\end{array} \qquad (73)$$

These simultaneous equations will have a nontrivial solution only if the determinant of the coefficients equals zero. Thus

$$(D_1B^2+\Sigma_1)(D_1B^2+\Sigma_1)(\ldots)(D_nB^2+\Sigma_n)-(K\Sigma_n)(\Sigma_1)(\Sigma_1)(\ldots) = 0 \tag{74}$$

or dividing through by the product $(\Sigma_n)(\Sigma_1)(\Sigma_1)(\ldots)$

$$\left(\frac{D_1}{\Sigma_1}B^2 + 1\right)\left(\frac{D_1}{\Sigma_1}B^2 + 1\right)\left(\ldots\right)\left(\frac{D_n}{\Sigma_n}B^2 + 1\right) - K = 0 \tag{75}$$

Substitution of L^2 for $\frac{D}{\Sigma}$ in Eq. (75) and rearranging

$$\frac{K}{(L_1^2B^2+1)(L_1^2B^2+1)(\ldots)(L_n^2B^2+1)} = 1 \tag{76}$$

Performing the indicated multiplication in the denominator of the left side of Eq. (76) and neglecting all terms higher than the second order

$$\frac{K}{1 + B^2(L_1^2 + L_1^2 + \ldots + L_n^2)} = 1 \tag{77}$$

Solving Eq. (77) for B^2 yields

$$B^2 = \frac{K - 1}{(L_1^2 + L_1^2 + \ldots + L_n^2)} \tag{78}$$

Substitution of Eqs. (51) and (53) into Eq. (78) gives

$$\left(\frac{\pi}{R}\right)^2 = \frac{\epsilon(\nu\frac{\Sigma_f}{\Sigma_a}) \left\{\exp\left[-\frac{\text{const.}}{\bar{\xi}}\left(\frac{\Sigma_a^{th}}{\Sigma_s}R\right)^{\text{const.}}\right]\right\} - 1}{L_1^2 + L_1^2 + \ldots + L_n^2} \tag{79}$$

The similarity between Eqs. (79) and (54) should be noted; in fact, if the multi-group energy increments are equal to those used in the Flügge-Tittle age calculation, then Eqs. (79) and (54) are almost exactly equal. As the energy region is divided into smaller energy groups the accuracy of the multi-group diffusion theory surpasses that of the modified single group theory.

B. Selection of Multi-Group Pi Terms

A suggested list of Pi terms for application in multi-group theory is listed in Table 8.

Table 8. Multi-group Pi terms

π_1	ϵ
π_2	ν
π_3	$\bar{\eta}$
π_4	$\frac{\Sigma_f}{\Sigma_a}$
π_5	$\frac{\Sigma_{s_{th}}}{\Sigma_a}$
π_6	$\frac{\Sigma_f}{\Sigma_{s_{th}}}$
π_7	$\tau\Sigma_a$
π_8	$\frac{L_1}{R}$
π_9	$\frac{L_2}{R}$
...	...
π_n	$\frac{L_n}{R}$

Although several alternate arrangements were possible, the Pi terms in Table 8 were chosen to allow the maximum degree of flexibility when matching the terms of a scale model to the prototype. Since a multi-group solution requires that the L values be equal for the model and prototype in each energy increment, it is visualized that a large number of calculations would be required to equalize the Pi terms in the model to those in the prototype. It would be a formidable task to obtain a solution to this problem without the aid of a computer.

VII. SUMMARY AND CONCLUSIONS

The application of dimensional analysis to homogeneous reactor design was severely restricted because all of the independent quantities have the same dimension (length) or are dimensionless. It is possible to develop a model design theory which is valid for both single and multiple group reactor theories by grouping the parameters into Pi terms. Matching the Pi terms in a multi-group problem was shown to be difficult.

A reduction in the size of a reactor model is not possible if both model and prototype contain identical materials in the same ratio and are both operated at the same temperature. Using a trial and error method to determine the composition of the model, it is possible to maintain Pi term equality when designing a model which is of a different size than the prototype.

Normally, the maximum attainable core radius ratio ($\frac{R_m}{R}$) was calculated to be 2.0 for enlarged models and 0.5 for reduced models; however, the typical values of the radius ratio varied between 1.5 to 1.75 and 0.57 to 0.67 for enlarged and reduced model sizes respectively. The volume ratio ($\frac{V_m}{V}$) for the three reactor shapes investigated (sphere, cylinder, rectangle) was shown to be the cube of the ($\frac{\Sigma_a}{\Sigma_{a_m}}$) ratio. Realistic volume ratios equaled approximately 2.6 for $V_m > V$ and 0.4 for $V_m < V$.

The Flügge-Tittle method of age determination was modified to enable its application to materials containing small percentages of hydrogen. It was shown that the accuracy of this method compares favorably with experimentally determined values of τ for various material mixtures.

The flexibility of the model design method was demonstrated by the example reactor problem. Theoretically it was possible to design a critical model of the prototype by appropriately distorting $\Sigma_{a_R}^{th}$, $\bar{\xi}$, D and τ. No attempt was made to determine if the combinations of materials used in the model reactor were compatible.

VIII. SUGGESTIONS FOR FUTURE INVESTIGATIONS

It is believed that this method of model design can be extended to include heterogeneous reactors. Because the highly absorbent fuel is concentrated in rods and plates in a heterogeneous reactor, the average flux developed in the fuel is lower than in the moderator. Three of the factors of the infinite multiplication factor must be modified to account for this flux level difference. The flux level difference in the resonance region causes the modification of p to [1, p. 263]

$$p(E) = \exp\left[-\frac{N_u}{\xi \Sigma_s}\cdot\frac{V_u \bar{\phi}_u}{V_m \bar{\phi}_m}\int_E^{E_o}\left(\frac{\Sigma_s}{\Sigma_s + \Sigma_{a_R}^{th}}\right)\sigma_{a_o}\frac{dE}{E}\right] \quad (80)$$

where V_u and V_m = volume of fuel and moderating material respectively

$\bar{\phi}_u$ and $\bar{\phi}_m$ = average resonance region flux existing in fuel and moderator respectively

The effective resonance integral in Eq. (80) is calculated using the empirical equations of the form [1, p. 258; 3, pp. 661-63]

$$RI = A + B\frac{S}{M} \quad (81a)$$

or

$$RI = A' + B'\left(\frac{S}{M}\right)^{\frac{1}{2}} \quad (81b)$$

where A and A' = volume absorption terms which are proportional to the atoms of fissionable material

per cubic centimeter of fuel

B and B' = surface absorption proportionality factor

S = surface area of the fuel in cm^2

M = mass of fuel in grams

Substituting Eqs. (81a) and (81b) into Eq. (80) gives

$$p = \exp\left\{-\frac{N_u}{\xi\Sigma_s} \cdot \frac{V_u\bar{\phi}_u}{V_m\bar{\phi}_m}\left[\text{const.} + \text{const.}\left(\frac{S}{M}\right)^{\text{const.}}\right]\right\} \tag{82}$$

The thermal utilization for a heterogeneous reactor is given by [3, p. 615]

$$f = \frac{\Sigma_{a_u} V_u \bar{\phi}_u}{\Sigma_{a_u} V_u \bar{\phi}_u + \Sigma_{a_m} V_m \bar{\phi}_m} \tag{83}$$

rearranging

$$f = \frac{1}{1 + \dfrac{\Sigma_{a_m} V_m \bar{\phi}_m}{\Sigma_{a_u} V_u \bar{\phi}_u}} \tag{84}$$

The value of the fast fission factor can be approximately calculated by [2, pp. 95-98; 1, p. 276]

$$\epsilon - 1 = \frac{(\sigma_f \nu - \sigma_f - \sigma_c)p}{\sigma - (\sigma_f \nu - \sigma_s)p} \tag{85}$$

where σ_f, σ_s, and σ = fission, elastic scattering and total cross sections for fast neutrons in the fuel

σ_c = cross section for nonfission capture of fast neutrons by the fuel

ν = number of fast neutrons produced per thermal fission

P = probability that a fission neutron generated from thermal fission will experience a collision in the fuel rod in which it is created

Approximate values of P have been calculated for slabs, cylinders and spheres as a function of $x\Sigma$ and $\frac{\Sigma_a}{\Sigma}$ [3, p. 712] where x equals the half-thickness of a slab, the radius of a cylinder or sphere, Σ equals the macroscopic transport cross section for the fuel and Σ_a equals $\frac{\Sigma_a}{D}$ for the fuel.

If dimensional analysis is applied to a heterogeneous reactor, the number of Π_i terms becomes large. For example, in the three quantities just examined, there are ten additional quantities (V, ϕ, V_m, ϕ_m, M, S, σ_c, σ, σ_a and P) and only one additional dimension (weight). If the total number of variables could be reduced, it may be possible to analyse heterogeneous reactors with this method.

The dimensional analysis method of designing a model may be utilized to study the shielding problems of proposed reactor designs.

Rearranging Eq. (9) gives

$$1 = \frac{Ke^{-B^2\tau}}{1 + \frac{DB^2}{\Sigma_a}} \qquad (86)$$

Substituting $\left(\frac{\pi}{R}\right)^2$ for B^2

$$1 = \frac{K_\infty}{1 + \frac{D}{\Sigma_a}\left(\frac{\pi}{R}\right)^2 - \left(\frac{\pi}{R}\right)^2 \tau} \tag{87}$$

Observe that the quantities $\frac{\tau}{R^2}$ and $\frac{D}{\Sigma_a R^2}$ are merely combinations of the Pi terms listed in Table 3.

$$\frac{\tau}{R^2} = \frac{\left(\frac{\tau}{D^2}\right)(\Sigma_a D)^2 \left(\frac{\Sigma_a^{th}}{\Sigma_a}\right)_R \left(\frac{\Sigma_f}{\Sigma_a^{th}}\right)^2}{\left(\frac{\Sigma_f}{\Sigma_a}\right)^2 (R\Sigma_a)^2} \tag{88}$$

substituting the appropriate Pi terms from Table 3 into Eq. (88) gives

$$\frac{\tau}{R^2} = \frac{\pi_9 \pi_8^2 \pi_6^2 \pi_7^2}{\pi_5^2 \pi_1^2} \tag{89}$$

$$\frac{D}{\Sigma_a R^2} = \frac{\left(\frac{\tau}{R^2}\right)\left(\frac{\Sigma_f}{\Sigma_a}\right)}{\left(\frac{\tau}{D^2}\right)(\Sigma_a D)\left(\frac{\Sigma_a^{th}}{\Sigma_a}\right)_R \left(\frac{\Sigma_f}{\Sigma_a^{th}}\right)} \tag{90}$$

by substitution of the Pi terms Eq. (90) reduces to

$$\frac{D}{\Sigma_a R^2} = \frac{\pi_9 \pi_8^2 \pi_6^2 \pi_7^2 \pi_5}{\pi_5^2 \pi_1^2 \pi_9 \pi_8 \pi_6 \pi_7} \tag{91}$$

or simplifying

$$\frac{D}{\Sigma_a R^2} = \frac{\pi_2 \pi_6 \pi_7}{\pi_5 \pi_1} \tag{92}$$

If

$$\frac{\tau}{R^2} = \frac{\tau_m}{R_m^2}$$

$$\frac{D}{\Sigma_a R^2} = \frac{D_m}{\Sigma_{a_m} R_m^2} \tag{93a}$$

then the fast and thermal neutron leakage fractions are equal for the model and the prototype. If the neutron multiplication factors are equal,

$$K = K_m \tag{93b}$$

then the magnitude of neutrons leaking from the model and the prototype should be equal for both fast and thermal neutrons.

It is believed that the shielding requirements for a large reactor could be determined and optimized on a model reactor.

It is suggested that this method of reactor design could be extended to reflected reactors. The reflector savings (δ) is given by [1, p. 235]

$$\delta = \frac{D_c}{D_r} L_r \tanh \frac{T}{L_r} \tag{94}$$

where D_c = diffusion coefficient for the reactor core

D_r = diffusion coefficient for the reflector

$L_r = \frac{D_r}{\Sigma_{a_r}}$ = slowing down length of the reflector

T = reflector thickness

There are three additional quantities (D_r, Σ_{a_r} and T) and no additional dimensions. Possible additional Pi terms which might be used to examine a reflected reactor are:

$$\pi_{r_1} = T\Sigma_r \tag{93}$$

$$\pi_{r_2} = D\Sigma_r$$

$$\pi_{r_3} = \frac{T}{R}$$

where R = reactor radius with reflector

Verification of the Modified Flügge-Tittle method of determining neutron age by experimentation is recommended before this method is used in any design calculations.

IX. LITERATURE CITED

1. Glasstone, S. and Edlund, M. C. Elements of nuclear reactor theory. Princeton, N. J., D. Van Nostrand Co., Inc. 1952.

2. Murray, R. L. Nuclear reactor physics. Englewood Cliffs, N. J., Prentice-Hall, Inc. 1957.

3. Weinberg, A. M. and Wigner, E. P. Physical theory of neutron chain reactors. Chicago, Ill., University of Chicago Press. 1958.

4. Weinberg, A. M. Pile neutron physics. U. S. Atomic Energy Commission Report M-3770. [Technical Information Service, AEC] [ca. 1946].

5. Lane, J. A., MacPherson, H. G. and Maslan, F. Fluid fuel reactors. Reading, Mass., Addison-Wesley Publishing Co., Inc. 1958.

6. Glasstone, S. Principles of nuclear reactor engineering. Princeton, N. J., D. Van Nostrand Co., Inc. 1955.

7. McMurry, H. L. Age in beryllium-water mixtures. U. S. Atomic Energy Commission Report IDO-16067. [Idaho Operation Office, AEC] 1953.

8. Ogburn, G. H. Measurement of neutron distribution in hydrogenous materials. U. S. Atomic Energy Commission Report NP-5869. [Technical Information Service, AEC] 1955.

9. Roberts, L. D. and Fitch, T. E. Slowing down distribution to indium resonance of fission neutrons from a point fission source in two mixtures of aluminum and light water, 1:1 and 1:2 by volume. U. S. Atomic Energy Commission Report ORNL-294. [Oak Ridge National Laboratory, Tenn.] 1949.

10. Reier, M. Age of polonium-beryllium and polonium-boron neutrons in various metal-water mixtures--a comparison between theory and experiment. Second United Nations International Conference on the Peaceful Uses of Atomic Energy, Geneva, Paper No. 635. 1958.

11. Wade, J. W. Neutron age in mixtures of D_2O and H_2O. Nucl. Sci. and Eng. Vol. 4, No. 1:12-24. July, 1958.

12. Murphy, Glenn. Similitude in engineering. New York, N. Y., Ronald Press Co. 1950.

13. Davis, M. V. and Hauser, D. T. Thermal-neutron data for the elements. Nucleonics. 16:87-89. March, 1958.

14. Hughes, D. J. and Harvey, J. A. Neutron cross sections. U. S. Atomic Energy Commission Report BNL-325. 2nd ed. [Brookhaven National Laboratory, Upton, N. Y.] 1958.

X. ACKNOWLEDGMENTS

Sincere appreciation is extended to Dr. Glenn Murphy, Head of the Department of Theoretical and Applied Mechanics, for his many valuable suggestions, prolonged encouragement and continued support throughout the investigation.

Recognition is extended to the members of the Nuclear Engineering Groups I and III of the Ames Laboratory for their assistance.

XI. APPENDIX

A. Derivation of Age Equation for Nuclei Mixtures

By definition the neutron age for a pure substance is given by

$$\tau(u) = \int_0^u \frac{D}{\bar{\xi}\Sigma_s}\,du \tag{96}$$

where u equals lethargy

Since

$$du = -\frac{dE}{E} \tag{97}$$

and

$$D = \frac{1}{\Sigma_s(1-\bar{\mu})} \tag{98}$$

Substitution of Eqs. (97) and (98) into Eq. (96) yields

$$\tau(E) = \int_E^{E_0} \frac{\frac{dE}{E}}{(\bar{\xi}\Sigma_s)\,[3\Sigma_s(1-\bar{\mu})]} \tag{99}$$

For a mixture of nuclei

$$\bar{\xi} = \frac{\sum_{i=1}^{n} \Sigma_{s_i}\bar{\xi}_i}{\Sigma_s} \tag{100}$$

and

$$\Sigma_s(1-\bar{\mu}) = \sum_{i=1}^{n} \Sigma_{s_i}\,(1-\bar{\mu}_i) \tag{101}$$

Substitution of Eqs. (100) and (101) into Eq. (99) gives the age for a mixture of nuclei as a function of energy.

$$\tau(E) = \int_E^{E_0} \frac{\frac{dE}{E}}{3\sum_{i=1}^{n} \Sigma_{s_i}\bar{\xi}_i \sum_{i=1}^{n} \Sigma_{s_i}(1-\bar{\mu}_i)} \tag{102}$$