Optimal and sub-optimal, low-thrust, Earth-Moon trajectories using sequential quadratic programming

by

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CHAPTER 1. INTRODUCTION

Historical Background

The National Aeronautical Space Administration (NASA) is currently exhibiting a renewed interest in low-thrust manned space missions. The Space Exploration Initiative (SEI) is a long-term project with goals including a manned exploration of Mars and the establishment of a permanent manned lunar base. Such a permanent lunar colony would require an inexpensive, efficient means of transporting supplies and possibly raw materials between the Earth and Moon. Low-thrust ion propulsion systems could transport greater payload fractions at a lower cost than conventional high-thrust solid or liquid chemical propulsion systems and would therefore be the prime candidate for such lunar missions.

Much of the interest in low-thrust propulsion has involved applications for Earth orbit transfer vehicles ([1],[2]). Recently, Aston [3] has investigated the merits and feasibility of using ion propulsion to ferry cargo between low Earth orbit and low lunar orbit. However, the volume of published work on computing the trajectory for a low-thrust, Earth-Moon mission appears to be somewhat limited. Early preliminary studies were performed by London [4] and Stuhlinger [5] while recent work on optimal Earth-Moon trajectories has been completed by Horsewood, Suskin and Pines [6] and Golan and Breakwell [7]. The previously published material on Earth-Moon trajectories impose assumptions and methods which differ from the work presented in this thesis.

Problem Statement

The objective of this study is to calculate the optimal trajectory between a circular, low-Earth, parking orbit and a circular, low-Moon, parking orbit. The optimal trajectory will be the trajectory which requires the minimum amount of fuel to complete the transfer from the Earth parking orbit to the Moon parking orbit. Only the planar transfer between parking orbits is considered; therefore, transfer from the Moon parking orbit to the lunar surface will not be addressed, and the return trip from the Moon orbit to the Earth orbit will not be investigated. The thrust level and propellant mass flow rate are both assumed to be constant. Therefore, the problem becomes one of minimizing the total engine-on time during the Earth-to-Moon transfer. The trajectory design variables will include the engine on/off switching times, the thrust direction time histories during thrusting arcs, and the geometry between the Earth and Moon at the initiation of the transfer.

CHAPTER 2. SYSTEM MODELS

The computation of low-thrust, Earth-Moon trajectories requires the numerical integration of the governing equations of motion. The optimal and sub-optimal trajectories include equations of motion governed by two-body or central force gravitational field dynamics and classical restricted three-body dynamics. This chapter presents the different dynamic systems and the resulting governing equations. The spacecraft and propulsion system models are also presented in this chapter.

Two-Body Dynamics

A large segment of the total Earth-Moon trajectory is in close proximity to both primaries since the total trajectory begins and ends in low circular orbits about the primaries and the low-thrust propulsion system can only slowly change the spacecraft's orbit. The continuous thrust escape and capture trajectories require long time durations to change the radial distance from the Earth or Moon, respectively. Therefore, two-body or central inverse-square gravitational field dynamics are used to approximate the trajectory in close proximity to either primary in order to reduce the computational effort associated with the numerical integration of the equations of motion. The two-body dynamics describe the motion of the spacecraft influenced only by the gravitational force directed toward the center of the primary attracting

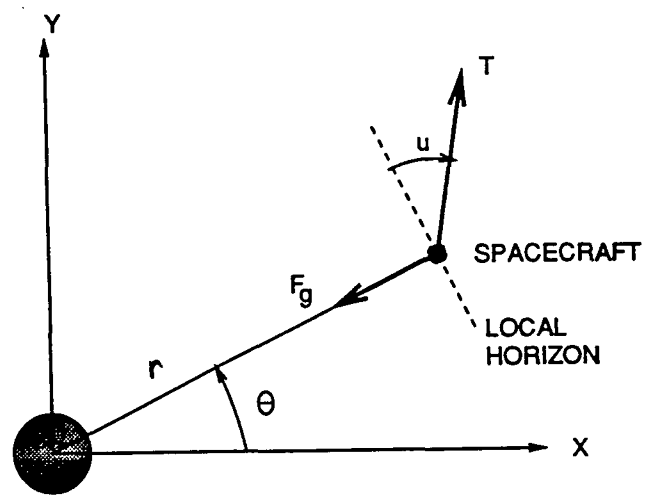

Figure 2.1: Thrusting Spacecraft in Body-Centered Polar Coordinates

body. The equations of motion based on two-body dynamics, which will be presented later, provide a very good approximation of the trajectory near a primary and will be utilized to solve the sub-optimal control problem.

Figure 2.1 shows the spacecraft in a polar coordinate system acted on by two forces: the thrust, T, and the gravitational force, F_g. The thrust direction angle, u, is measured from the local horizon and is considered positive above the horizon in the direction of the spacecraft's motion.

The radial and circumferential components of the inertial acceleration in rotating polar coordinates [8] are:

$$a_r = \ddot{r} - r\dot{\theta}^2 \tag{2.1}$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{2.2}$$

The radial and circumferential components of the total forces on the spacecraft are as follows:

$$F_r = -F_g + T\sin u \tag{2.3}$$

$$F_\theta = T\cos u \tag{2.4}$$

The gravitational force, F_g, is governed by Newton's Universal Law of Gravitation [9] and can be written as:

$$F_g = \frac{GMm}{r^2} \tag{2.5}$$

where

G : Universal gravitational constant
M : mass of the attracting body (Earth or Moon)
m : mass of the particle in motion (spacecraft)

The acceleration and force components can be equated using Newton's second law of motion:

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{GMm}{r^2} + T\sin u \tag{2.6}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = T\cos u \tag{2.7}$$

The radial and circumferential acceleration of the spacecraft can be obtained by dividing equations (2.6) and (2.7) by the mass of the spacecraft:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2} + a_T\sin u \tag{2.8}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_T\cos u \tag{2.9}$$

Here, the thrust acceleration of the spacecraft, a_T, is computed by dividing the constant thrust of the propulsion system by the mass of the spacecraft:

$$a_T(t) = \frac{T}{m_0 - \dot{m}t} \tag{2.10}$$

where m_0 and $\dot{m}$ are the initial mass of the spacecraft and the constant propellant mass flow, respectively. The thrust acceleration a_T is variable and steadily increases with time when the propulsion system is operating since the mass of the spacecraft decreases.

Equations (2.8) and (2.9) are the governing equations of motion for a thrusting spacecraft in a central inverse-square gravity field. This fourth-order system can be reduced to four first-order differential equations by introducing the following state variables:

$x_1 = r$ (radial position)
$x_2 = \dot{r}$ (radial velocity)
$x_3 = r\dot{\theta}$ (circumferential velocity)
$x_4 = \theta$ (polar angle)

The resulting first-order equations of motion become:

$$\dot{x}_1 = x_2 \tag{2.11}$$

$$\dot{x}_2 = \frac{x_3^2}{x_1} - \frac{GM}{x_1^2} + a_T \sin u \tag{2.12}$$

$$\dot{x}_3 = -\frac{x_2 x_3}{x_1} + a_T \cos u \tag{2.13}$$

$$\dot{x}_4 = \frac{x_3}{x_1} \tag{2.14}$$

The fourth state equation, equation (2.14), is uncoupled from the system but is included to allow tracking of the spacecraft's angular position. These four equations are numerically integrated to describe the motion of a thrusting spacecraft in a central gravity field. Equation (2.10) is a simple linear relation required to compute the current thrust acceleration of the variable mass spacecraft during the numerical integration of the four state equations.

Three-Body Dynamics

The dynamics of the spacecraft along an Earth-Moon trajectory are studied in the context of the classical restricted three-body problem which is defined by two bodies revolving with a constant angular rate in circular orbits about their common center of mass. The two bodies are termed the primaries and are considered to be point masses whose motion is completely determined by their mututal gravitational attraction. A third body, in this case the spacecraft, moves in the plane defined by the two revolving primaries and is considered to be of negligible mass in comparison to the primaries and therefore does not influence the motion of the primaries. The restricted three-body problem describes the motion of the third body. The restricted three-body problem provides an accurate model of the Earth-Moon system since the Moon's orbit about the common center of mass is nearly circular with an eccentricity of 0.05.

The equations of motion without the thrust terms for the restricted three-body problem were originally formulated in a rotating, Cartesian, coordinate system with the origin at the Earth-Moon system center of mass (or barycenter) as presented by Szebehely [10]. I then added the thrust acceleration terms to create a complete fourth-order set of governing equations of motion for powered flight. These were numerically integrated to determine the motion of the spacecraft. The restricted three-body problem dynamics were observed to be accurate and well behaved in the rotating, barycentered, Cartesian frame when the spacecraft was moving in a non-oscillatory motion at a substantial distance from either primary. However, when the spacecraft is in a close orbit around one of the primaries, a relatively small integration step size was required for accurate numerical integration of the equations of motion. This was attributed to the use of a Cartesian frame and the subsequent oscillations in position and velocity components as the spacecraft travels on a cyclical trajectory about the primary body. The numerically integrated equations of motion also produced inaccurate results when the spacecraft is in close proximity to the Moon since the origin of the coordinate system is very far from the spacecraft and therefore can only detect very small changes in the state vector.

These problems can be alleviated by integrating the restricted three-body problem dynamics in a body-centered, rotating polar coordinate system. The polar coordinate system eliminates the problem of oscillating state variables since the radial distance and polar angle change slowly with time and are monotonically increasing. The ability to set the coordinate system at either primary body provides more accuracy when the spacecraft is in vicinity of that primary.

The equations of motion are formulated in a body-centered, rotating polar coordinate system by starting with the restricted three-body problem dynamics in an Earth-centered, inertial Cartesian frame as presented by Egorov [11]:

$$\ddot{X} = -\frac{Gm_1X}{R_1^3} + \frac{Gm_2(X_M - X)}{R_2^3} - \frac{Gm_2X_M}{D^3} \tag{2.15}$$

$$\ddot{Y} = -\frac{Gm_1Y}{R_1^3} + \frac{Gm_2(Y_M - Y)}{R_2^3} - \frac{Gm_2Y_M}{D^3} \tag{2.16}$$

where

X : x-location of the spacecraft
Y : y-location of the spacecraft
X_M : x-location of the Moon
Y_M : y-location of the Moon
R_1 : Earth to spacecraft distance
R_2 : Moon to spacecraft distance
D : Earth to Moon distance

The Earth-centered frame is shown in Figure 2.2. These equations represent the absolute acceleration of a non-thrusting spacecraft influenced only by the gravitational forces of the two primaries in an Earth-centered, inertial frame.

The thrust acceleration components in the Earth-centered Cartesian frame are:

$$a_{T_x} = a_T \sin u \cos\theta - a_T \cos u \sin\theta \tag{2.17}$$

$$a_{T_y} = a_T \sin u \sin\theta + a_T \cos u \cos\theta \tag{2.18}$$

Again, a_T is the thrust acceleration magnitude computed by the linear relation during the numerical integration process:

$$a_T(t) = \frac{T}{m_0 - \dot{m}t} \tag{2.19}$$

The thrust direction angle, u, is measured with respect to the local Earth horizon as previously defined. The Earth-centered polar angle, θ, is measured from the posi-

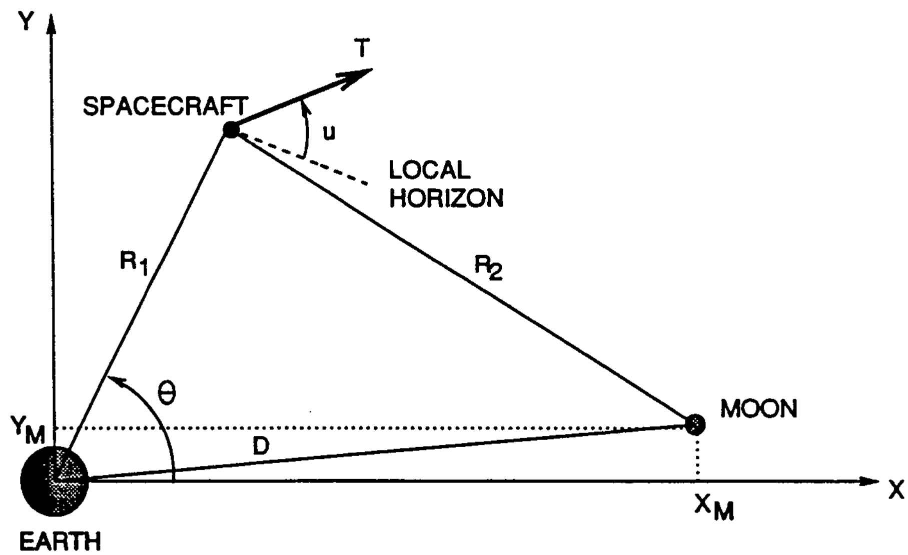

Figure 2.2: Thrusting Spacecraft in Earth-Centered, Inertial Cartesian Coordinates

tive x-axis to the radius vector from the Earth to the spacecraft as shown in Figure 2.2. The thrust acceleration components are added to the gravitational acceleration components from equations (2.16) and (2.17) to form the absolute acceleration of the spacecraft in the Earth-centered, inertial frame:

$$\ddot{X} = -\frac{Gm_1X}{R_1^3} + \frac{Gm_2(X_M - X)}{R_2^3} - \frac{Gm_2X_M}{D^3} + a_{T_x} \tag{2.21}$$

$$\ddot{Y} = -\frac{Gm_1Y}{R_1^3} + \frac{Gm_2(Y_M - Y)}{R_2^3} - \frac{Gm_2Y_M}{D^3} + a_{T_y} \tag{2.22}$$

The restricted three-body problem equations of motion can be referenced to a rotating, Earth-centered coordinate system by using relative motion analysis and accounting for the Coriolis and centripetal accelerations [8]. The accelerations relative to the rotating frame are as follows:

$$\ddot{\vec{r}}_{rot} = \ddot{\vec{r}}_{abs} - \dot{\vec{\omega}} \times \vec{r}_{rot} - \vec{\omega} \times \vec{\omega} \times \vec{r}_{rot} - 2\vec{\omega} \times \dot{\vec{r}}_{rot} \tag{2.22}$$

After performing the vector products, equation (2.22) becomes:

$$\ddot{X}_{rot} = \ddot{X}_{abs} + 2\dot{Y}_{rot}\omega + Y_{rot}\dot{\omega} + X_{rot}\omega^2 \tag{2.23}$$

$$\ddot{Y}_{rot} = \ddot{Y}_{abs} - 2\dot{X}_{rot}\omega - X_{rot}\dot{\omega} + Y_{rot}\omega^2 \tag{2.24}$$

where

$\ddot{X}_{abs}, \ddot{Y}_{abs}$: absolute acceleration
$\ddot{X}_{rot}, \ddot{Y}_{rot}$: relative acceleration (with respect to the rotating frame)
$\dot{X}_{rot}, \dot{Y}_{rot}$: relative velocity (with respect to the rotating frame)
X_{rot}, Y_{rot} : relative position (with respect to the rotating frame)
ω : angular rate of the rotating frame
$\dot{\omega}$: angular acceleration of the rotating frame

The new Earth-centered frame is defined with the x-axis along the Earth-Moon line with the positive x-direction pointing away from the Moon. The frame rotates with a constant angular rate, ω, as required for the restricted three-body problem and therefore the terms including angular acceleration are not present. The components of the absolute acceleration, $\ddot{X}_{abs}$ and $\ddot{Y}_{abs}$, can be replaced by substituting equations (2.20) and (2.21), the absolute acceleration of the spacecraft due to gravitational and thrust forces. The coordinates of the Moon no longer need to be included in equations (2.20) and (2.21) since the x-axis rotates with the Earth-Moon line; therefore the x-coordinate of the Moon, X_M, is replaced by $-D$ and the Moon's y-coordinate, Y_M, is set to zero. The equations of motion with respect to the rotating frame can be rewritten, after dropping the subscripts, as follows:

$$\ddot{X} = -\frac{Gm_1X}{R_1^3} - \frac{Gm_2(X+D)}{R_2^3} + \frac{Gm_2D}{D^3} + a_{T_x} + 2\dot{Y}\omega + X\omega^2 \tag{2.25}$$

$$\ddot{Y} = -\frac{Gm_1Y}{R_1^3} - \frac{Gm_2Y}{R_2^3} + a_{T_y} - 2\dot{X}\omega + Y\omega^2 \tag{2.26}$$

This fourth-order system of equations can be made dimensionless by defining the reference unit length as the mean Earth-Moon distance, the reference unit time as the inverse of the angular rate of the Earth-Moon system, and the reference unit mass as the total mass of the Earth-Moon system:

$$x = \frac{X}{D} \ , \ y = \frac{Y}{D} \tag{2.27}$$

$$\tilde{t} = \omega t \tag{2.28}$$

$$\tilde{m} = \frac{m}{M} \tag{2.29}$$

where

X, Y, m, t	:	dimensional length, mass and time
$x, y, \tilde{m}, \tilde{t}$	:	dimensionless length, mass and time
D	:	Earth to Moon distance
M	:	Total mass of the Earth-Moon system
ω	:	Angular rate of the Earth-Moon system

The dimensionless equations of motion for the restricted three-body problem with thrust terms in an Earth-centered, rotating, Cartesian frame are:

$$\ddot{x} = -\frac{\mu_1 x}{r_1^3} - \frac{\mu_2(x+1)}{r_2^3} + \mu_2 + \tilde{a}_{T_x} + 2\dot{y} + x \tag{2.30}$$

$$\ddot{y} = -\frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3} + \tilde{a}_{T_y} - 2\dot{x} + y \tag{2.31}$$

The parameters $\tilde{a}_{T_x}$ and $\tilde{a}_{T_y}$ are the dimensionless thrust acceleration components. The parameters μ_1 and μ_2 correspond to the dimensionless gravitational parameters of the primaries.

The objective is to formulate the restricted three-body body problem with thrust terms in a rotating, body-centered, polar coordinate system for increased accuracy and flexibility for numerical integration. The dimensionless radial and circumferential accelerations in rotating, Earth-centered, polar coordinates are calculated using equations (2.30) and (2.31) and the Earth-centered polar angle, θ, as follows:

$$\tilde{a}_r = \ddot{x}\cos\theta + \ddot{y}\sin\theta \tag{2.32}$$

$$\tilde{a}_\theta = -\ddot{x}\sin\theta + \ddot{y}\cos\theta \tag{2.33}$$

Equations (2.32) and (2.33) express the dimensionless radial and circumferential acceleration of the spacecraft due to the restricted three-body problem dynamics, the low-thrust propulsion system, and the transformation from an inertial frame to a rotating frame. The kinematic equations of a particle's acceleration in polar coordinates, equations (2.1) and (2.2), can be equated to the dimensionless acceleration components, equations (2.32) and (2.33) as shown below:

$$\ddot{r} - r\dot{\theta}^2 = \tilde{a}_r \tag{2.34}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \tilde{a}_\theta \tag{2.35}$$

This fourth-order system can be reduced to four first-order differential equations by introducing the state variables:

$x_1 = r$ (dimensionless Earth-spacecraft position)
$x_2 = \dot{r}$ (dimensionless radial velocity in rotating frame)
$x_3 = r\dot{\theta}$ (dimensionless circumferential velocity in rotating frame)
$x_4 = \theta$ (polar angle)

The resulting first-order equations of motion are:

$$\dot{x}_1 = x_2 \tag{2.36}$$

$$\dot{x}_2 = \frac{x_3^2}{x_1} + \tilde{a}_r \tag{2.37}$$

$$\dot{x}_3 = -\frac{x_2 x_3}{x_1} + \tilde{a}_\theta \tag{2.38}$$

$$\dot{x}_4 = \frac{x_3}{x_1} \tag{2.39}$$

The fourth state equation is uncoupled from the system but is required for the transformation from the Cartesian coordinates to the polar coordinates. These four equations are numerically integrated to determine the motion of a thrusting spacecraft in the context of the restricted three-body problem.

The derivation of the complete equations of motion for the restricted three-body problem in a Moon-centered, rotating, Cartesian frame are shown in Appendix A and the resulting dimensionless equations are:

$$\ddot{x} = -\frac{\mu_1(x-1)}{r_1^3} - \frac{\mu_2 x}{r_2^3} - \mu_1 + \tilde{a}_{T_x} + 2\dot{y} + x \tag{2.40}$$

$$\ddot{y} = -\frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3} + \tilde{a}_{T_y} - 2\dot{x} + y \tag{2.41}$$

Equations (2.40) and (2.41) are referenced to a rotating coordinate system centered at the Moon; therefore x and y are the position coordinates of the spacecraft with respect to the center of the Moon. The Moon-centered accelerations are then transformed and numerically integrated in a rotating polar frame as previously described for the Earth-centered frame and as detailed in Appendix A.

Since the restricted three-body problem equations of motion are defined in a rotating frame, the initial inertial velocities (orbital velocities) must be transferred to the respective body-centered rotating frame. The transformation is performed using relative motion analysis [8] and accounting for the rotation of the coordinate frame:

$$\dot{\vec{r}}_{rot} = \dot{\vec{r}}_{abs} - \vec{\omega} \times \vec{r}_{rot} \tag{2.42}$$

The radial and circumferential velocities relative to the rotating system are:

$$\dot{r}_{rot} = \dot{r}_{abs} \tag{2.43}$$

$$r\dot{\theta}_{rot} = r\dot{\theta}_{abs} - \omega r \tag{2.44}$$

Finally, the restricted three-body problem dynamics were numerically tested in both rotating polar coordinate frames by initializing the spacecraft at the Lagrangian point L_4. There are five Lagrangian points in the restricted three-body problem where the resultant force is zero. Therefore a particle with zero relative velocity will remain fixed in the rotating Earth-Moon coordinate system. The Lagrangian points L_4 and L_5 form two equilateral triangles with the two primaries at the verticies. The spacecraft was initialized at the L_4 point with zero thrust and zero velocity relative to the rotating, body-centered frame. Numerical integration of the restricted three-body problem dynamics in both the Earth-centered and Moon-centered rotating polar coordinate systems showed that the spacecraft remain fixed in the rotating frame and therefore verified the equations of motion for this case.

Spacecraft Model

The low-thrust ion propulsion system is modeled by a thrust-limited system which implies operation at one fixed thrust magnitude and one fixed propellant mass flow rate. The system is assumed to be operate without any thrust build-up transient time or warm-up time.

The ion propulsion system and spacecraft parameters are a combination of the vehicle parameters used by London [4] and Stuhlinger [5]. Both authors independently investigated Earth-Moon trajectories using low-thrust propulsion. London developed Earth-Moon trajectories for a range of thrust-to-weight (T/W) ratios. The initial thrust-to-weight ratio for this study is $3(10^{-3})$ which corresponds to the highest T/W ratio used by London. The initial total mass of the spacecraft in low Earth orbit is 100,000 kg which corresponds to the spacecraft mass used in Stuhlinger's investigation of a low-thrust Earth-Moon ferry system. The constant thrust magnitude can be calculated using the initial T/W ratio and the initial mass in Earth orbit.

The constant propellant mass flow rate, $\dot{m}$, is computed using London's fuel requirements for a round-trip Earth-Moon mission with $T/W = 3(10^{-3})$ as follows:

$$\dot{m} = \frac{fuel\ mass\ required}{total\ engine - on\ time} \tag{2.45}$$

The resulting mass flow rate is 0.03 kg/sec.

The specific impulse, I_{sp}, of the ion propulsion system is calculated from the constant thrust magnitude and constant propellant flow rate as shown below [5]:

$$I_{sp} = \frac{T}{g\dot{m}} \tag{2.46}$$

where g is the Earth gravitational acceleration at sea level. The specific impulse is 10,047 sec which corresponds to London's I_{sp} value for $T/W = 3(10^{-3})$. The ion propulsion system parameters and spacecraft parameters are summarized in Table 2.1.

Table 2.1: Ion Propulsion System and Spacecraft Characteristics

Initial Spacecraft mass in Low Earth Orbit	$100,000\ kg$
Thrust of Ion Engine	$2,942\ N$
Thruster Efficiency	1.0
Specific Impulse	$10,047\ sec$
Propellant Exhaust Velocity	$98.52\ km/sec$
Propellant Flow Rate	$0.0299\ kg/sec$
Initial Thrust-to-Weight Ratio	$3(10^{-3})$
Power Requirement	$144.9\ (10^6)\ Watt$

The T/W ratio used for this study is at the upper end of the low-thrust propulsion system spectrum and may be actually classified as a "high" low-thrust system. The input power requirement for such a system is computed from [5]:

$$P = \frac{\dot{m}c^2}{2\eta} \tag{2.47}$$

where η is the thruster efficiency and c is the exhaust velocity of the ion propulsion system. The exhaust velocity is computed from the constant thrust and mass flow rate [5]:

$$c = \frac{T}{\dot{m}} \tag{2.48}$$

The resulting power requirement assuming ideal conditions ($\eta = 1$) is 145 MW. The projected power and I_{sp} requirements for an ion propulsion system capable of Moon and Mars manned missions are 100 kW to 1 MW and 3000 to 10,000 sec, respectively [12]. Therefore, the power requirements for the spacecraft model in this study are outside the technology realm projected by Reference [12]. However, this is not a serious drawback since this study presents a methodology for solving optimal, low-thrust, Earth-Moon trajectories which can be applied to a range of T/W ratios. This paper presents numerical results for only one initial thrust-to-weight ratio, namely $T/W = 3(10^{-3})$. This "high" low-thrust level is chosen to keep the total trip time relatively low and therefore to reduce computational costs. For example, Stuhlinger shows that a low-thrust spacecraft with $T/W = 1.2(10^{-4})$ requires 80 days of propulsion time to reach a low Moon orbit while London shows that a spacecraft with $T/W = 3(10^{-3})$ requires 2.6 days of propulsion time to complete the same trip. The solution method presented in this paper can eventually be applied to compute an optimal Earth-Moon trajectory with a "moderate" low-thrust level; namely $T/W = 1(10^{-4})$.

CHAPTER 3. SOLUTION METHOD

This chapter discusses the numerical methods that are used to solve the optimal control problem. A systematic approach for solving the minimum fuel Earth-Moon trajectory is also presented.

Direct Approach

A general problem statement for an optimal control problem is as follows.

Find the control time history $u(t)$, $t_0 \leq t \leq t_f$, which minimizes the performance index:

$$J = \phi[x(t_f)] + \int_{t_0}^{t_f} L(x, u, t)dt \tag{3.1}$$

subject to the differential state equation constraints

$$\dot{x} = f(x, u, t) \tag{3.2}$$

and the control inequality constraints

$$u_{min} \leq u(t) \leq u_{max} \tag{3.3}$$

and the initial and terminal contraints

$$x(t_0) = x_0 \ , \ x(t_f) = x_f \tag{3.4}$$

This class of optimal control problems can be solved via either a direct or indirect method. An indirect method solves the two-point boundary value problem that results from the application of the necessary conditions for optimality. A direct method iterates on the control $u(t)$ in an attempt to minimize the performance index. In this study, the minimum fuel, low-thrust, Earth-Moon trajectory will be solved via a direct method which replaces the optimal control problem with an approximate nonlinear programming problem. The problem then becomes a constrained parameter optimization problem with the infinite-dimensional control time history $u(t)$ replaced with a cubic spline fit through a finite set of control parameters. The performance index is then minimized over a fixed set of control parameters instead of a continuous, infinite-dimensional time history.

The control parameters for the minimum fuel Earth-Moon trajectory problem represent both discrete and continuous trajectory design variables. The design variables include engine on/off switch times in the former case and thrust direction time histories in the latter. The thrust direction time history is parameterized by a cubic spline interpolation through a fixed number of equally-spaced thrust direction angles as shown in Figure 3.1. The trajectory is computed by numerically integrating the equations of motion with the current set of control parameters, and the performance index is calculated explicitly during the numerical simulation. The control parameters are adjusted between iterations until some termination criteria approximating the necessary conditions is satisfied.

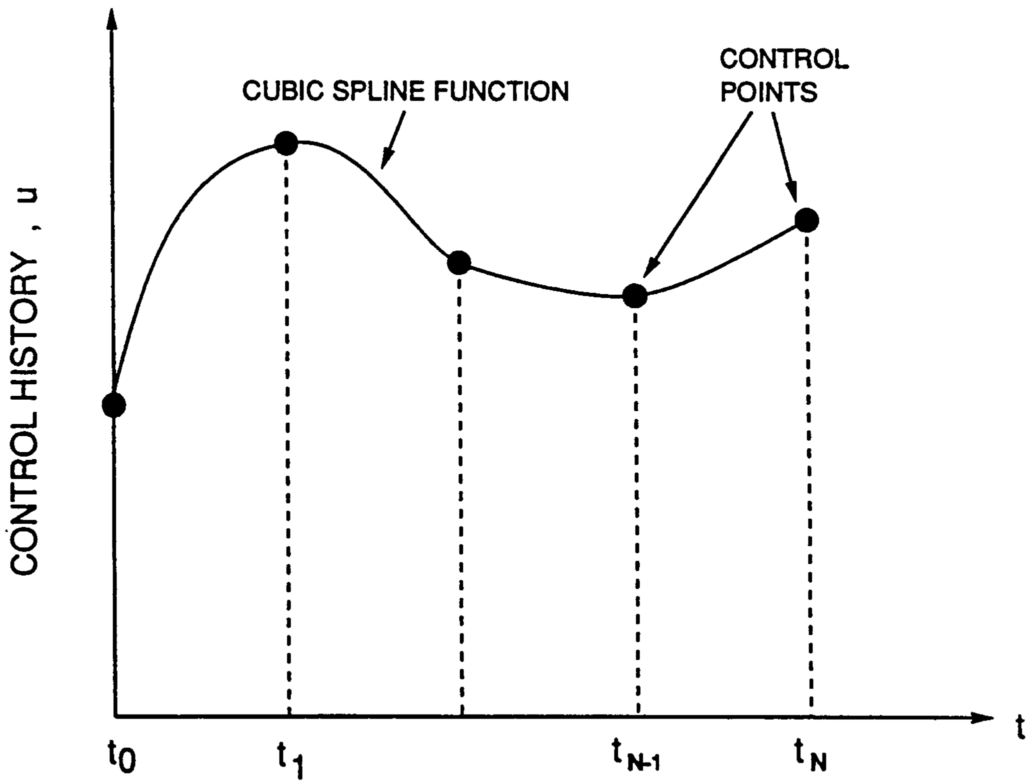

Figure 3.1: Discrete Parameterization of $u(t)$

Sequential Quadratic Programming

The approximate parameter optimization problem is solved using sequential quadratic programming. This numerical optimization method is chosen because of the flexibility and ease of implementation. Sequential quadratic programming requires a numerical simulation with explicit calculations of the performance index and constraints. Therefore, changes in the dynamic model, performance index and constraints can be made with relative ease. Unlike indirect methods, there are no restrictions to the forms of the control variables; therefore both continous and bang-bang control can be easily implemented.

The disadvantage of using sequential quadratic programming is that the solution is not as precise as the solution obtained via an indirect or variational method. Another disadvantage is that the form of the control must be assumed a priori; for example, the number of engine switch times must be specified in advance.

Sequential quadratic programming is a constrained Quasi-Newton method which exhibits superlinear convergence [13]. The method solves the approximate nonlinear programming problem by solving a sequence of related quadratic programming problems [14]. A quadratic programming problem is a parameter optimization problem with a quadratic performance index subject to linear constraints. Quadratic programming problems are well behaved, and several methods are available for their numerical solution. The solution of the quadratic programming problem is equivalent to the solution of the linearized necessary conditions for the nonlinear problem. Therefore, the approximate nonlinear programming problem is solved by solving a sequence of quadratic programming problems. A basic outline ([13], [14]) of the sequential quadratic programming algorithm is shown below.

i) Supply an initial guess for the control parameter vector and choose a positive definite Hessian matrix. Compute the partial derivatives of the performance index and constraints with respect to the control parameters via numerical integration and finite difference equations.

ii) Solve a quadratic programming problem for the corrections to the control parameter vector and the corresponding Lagrange multipliers. The correction vector is the direction of search.

iii) Calculate the step-size value by minimizing an auxiliary performance index along the direction of search. The step-size selection is used to force convergence from poor initial guesses of the control parameter vector.

iv) Update the control parameter vector with the step-size value and the resulting correction vector. Compute a new trajectory and test the termination criteria for convergence.

v) Update the Hessian matrix with a variable-metric scheme to insure that the matrix stays positive-definite.

Hierarchy of Problems

The minimum fuel, low-thrust, Earth-Moon trajectory is solved by formulating and successively solving a hierarchy of problems. Each problem is more detailed and difficult than the preceding problem and its solution provides valuable information for solving the next problem. The hierarchy of problems is classified into three basic phases:

i) The first set of problems solves optimal low-thrust escape and capture trajectories with respect to the Earth and Moon parking orbits. The objective is to maximize the energy at the end of a fixed period of continuous thrusting. The results will provide tractable boundary conditions for the next phase.

ii) The next problem solves a sub-optimal minimum fuel trajectory from the Earth parking orbit to the Moon parking orbit. The maximum energy Earth-escape and Moon-capture trajectory data supply the initial conditions and terminal constraints for this translunar phase.

iii) Finally, a full three-phase trajectory optimization problem starting at the Earth parking orbit and terminating at the Moon parking orbit is solved. Although this full simulation is computationally demanding, the sub-optimal solution from phase (ii) is used as an initial guess and therefore reduces the computational load and greatly increases the convergence properties.

CHAPTER 4. EARTH ESCAPE AND MOON CAPTURE OPTIMAL TRAJECTORIES

Numerical results for the three basic sets of problems are presented in the next three chapters. All numerical computations were performed on the Iowa State University NAS/9160 computer and the NASA Lewis Research Center VAX computer system using Fortran 77 with double precision arithmetic. A standard fourth-order, fixed-step, Runge-Kutta integration routine is used throughout the study to numerically integrate the equations of motion.

Optimal Earth escape and Moon capture trajectories with continuous thrust are computed in an attempt to remove these long duration spiral trajectories from the sub-optimal minimum fuel Earth-Moon trajectory problem. By solving a range of optimal escape and capture trajectories, velocity vectors and spiral times in the vicinity of the Earth and Moon can be computed by interpolating among the respective trajectory data. The velocity vectors will provide initial conditions near the Earth and terminal constraints near the Moon for the sub-optimal trajectory problem. Therefore, the sub-optimal translunar phase will not require simulation of the long duration Earth escape and Moon capture spiral trajectories.

Optimal Earth Escape Trajectories

Problem Statement

The optimal Earth escape trajectory problem may be stated as follows.

Find the thrust direction time history, $u(t)$, $t_0 \leq t \leq t_f$, for the optimal control problem:

Minimize :

$$J = -\frac{v(t_f)^2}{2} + \frac{\mu}{r(t_f)} \tag{4.1}$$

subject to

$$\dot{x}_1 = v_r = x_2 \tag{4.2}$$

$$\dot{x}_2 = \dot{v}_r = \frac{x_3^2}{x_1} - \frac{\mu}{x_1^2} + a_T \sin u \tag{4.3}$$

$$\dot{x}_3 = \dot{v}_\theta = -\frac{x_2 x_3}{x_1} + a_T \cos u \tag{4.4}$$

$$\dot{x}_4 = \dot{\theta} = \frac{x_3}{x_1} \tag{4.5}$$

and the boundary conditions

$$x_1(t_0) = r_0 \tag{4.6}$$

$$x_2(t_0) = 0 \tag{4.7}$$

$$x_3(t_0) = \sqrt{\frac{\mu}{r_0}} \tag{4.8}$$

$$x_4(t_0) = 0 \tag{4.9}$$

$$t_f = constant \tag{4.10}$$

The objective is to maximize the sum of the spacecraft's kinetic and potential energy at the end of a prescribed flight time under continuous thrust. The choice of maximum terminal energy as the performance index is based on engineering judgment. Whether or not the maximum energy trajectory ultimately corresponds to the minimum fuel trajectory for the total Earth-Moon mission is not important at this point; the results of these maximum energy trajectories will be used as boundary conditions for the sub-optimal minimum fuel problem to be presented later.

The motion of the spacecraft is governed by the equations of motion derived for a spacecraft with thrust in a central inverse-square gravity field. The spacecraft is initially in a circular orbit 315 km above the Earth. Two-body dynamics are used to reduce the complexity of the equations of motion and thus allow fewer integration steps for adequate accuracy. Several fixed end-time, fixed thrust direction trials were run with varying integration step sizes and the final radii and velocities for step sizes of 30 seconds and 20 minutes were observed to differ by less than 0.12%. Therefore, the two-body equations of motion are observed to be accurate for an integration step size of about 20 minutes, or roughly four and a half integration steps for every circular orbit. Since the optimal trajectory was observed to reach Earth escape conditions in about two days of continuous thrusting, 200 integration steps are used throughout for the optimal escape and capture trajectories. Two-body dynamics are also used to produce symmetric trajectories without regard to the position of the Moon at the initiation of the escape spiral. Therefore, the resulting optimal trajectories will be equivalent for all initial Earth polar angles. Finally, the chosen range of Earth escape and Moon capture trajectories remain sufficiently close to the primary body to allow accurate modeling of the spacecraft's motion with two-body dynamics.

The thrust direction time history, $u(t)$, is the only trajectory design variable for this fixed end-time problem and is parameterized by a set of equally-spaced discrete control points. The thrust direction time history is computed by fitting a cubic spline function through the control points. The initial guess for the thrust direction angle is set at all zero control points which results in thrusting along the horizontal direction for the first iteration.

A tradeoff exists between the accuracy of the optimal solution and the computational load for the selection of the number of control points. The maximum energy spiral has been solved for parameterizations of six, eleven and twenty-one control points and the results are shown in Table 4.1. Although the eleven and twenty-one control point parameterizations show substantial improvements in the performance index, they also exhibit an increase in computation time. Since the results of the maximum energy trajectories are to be used only for the sub-optimal solution, a six control point parameterization is used for the Earth escape trajectory.

Table 4.1: Number of Control Points Study

No. of Control Pts.	Iterations	CPU Time	Performance Improvement, %Base
6 (base)	22	1.93 *sec*	–
11	20	3.78 *sec*	8.3%
21	23	7.13 *sec*	10.9%

The maximum terminal energy trajectories are solved for a range of fixed end-times with the longest Earth spiral time chosen so that the resulting optimal trajectory just achieves escape conditions from the Earth's gravitational field. The initial thrust-to-weight ratio is sufficiently high enough to reach local Earth escape velocity at distance of 15.59 Earth radii after about two days of thrusting. A comparison with a maximum two-body energy trajectory using the restricted three-body problem dynamics and the same final end-time shows that the two final radii and the final velocities differ by less than 0.007%. Therefore, the two-body dynamics model produces very accurate trajectories from the parking orbit to Earth escape conditions.

Numerical Results

A typical convergence history for a maximum terminal energy trajectory with a fixed flight time of two days is shown in Table 4.2. The performance index (negative terminal energy), the normalized change in performance between iterations, and the norm of the Lagrangian gradient are presented. The sequential quadratic programming optimization method terminates when the constraints are satisfied and the when either the change in performance index between iterations or the norm of the Lagrangian gradient becomes smaller than the prescribed tolerances. The maximum energy problem is without terminal state constraints and the tolerances are set at 10^{-4} for all termination cases. The problem converged after 22 iterations when then change in performance between iterations became sufficiently small.

Table 4.2: Typical Convergence History

Iteration	J (Performance)	Δ J	Lagrangian Gradient Norm
1	$0.27060(10^{-2})$	–	–
2	$0.24964(10^{-2})$	$0.84(10^{-1})$	$0.15(10^{-1})$
3	$0.15735(10^{-2})$	0.59	$0.14(10^{-1})$
.	.	.	.
.	.	.	.
.	.	.	.
20	$-0.20578(10^{-2})$	$0.21(10^{-3})$	$0.24(10^{-3})$
21	$-0.20582(10^{-2})$	$0.20(10^{-3})$	$0.20(10^{-3})$
22	$-0.20584(10^{-2})$	$0.84(10^{-4})$	$0.19(10^{-3})$

The optimal escape spiral trajectory for a fixed flight time of two days is presented in the Cartesian frame as shown in Figure 4.1. This maximum energy trajectory achieves Earth escape conditions after nearly twelve revolutions about the Earth. The optimal thrust direction time history is presented in Figure 4.2 and the corresponding flight path angle time history is shown in Figure 4.3. The parameterization of the thrust direction time history is shown by the cubic spline function fit through the six optimal control points. The optimal thrust direction and flight path angle time histories exhibit the same general shape; both start near or at zero and slowly increase to a final value of about 39 degrees. Therefore, the thrust direction angle closely follows the flight path angle and therefore the velocity vector.

Sixteen maximum energy trajectories have been produced for a range of fixed end-times. The final velocity components and spiral times for the sixteen trials are plotted against the final trajectory radii as shown in Figures 4.4, 4.5, and 4.6. These curves are used to supply the sub-optimal translunar phase with tractable initial conditions. The resulting curves are smooth, and accurate velocity components and spiral times can be computed by fitting a variable-degree polynomial through the sixteen data points with final radial distance as the independent variable.

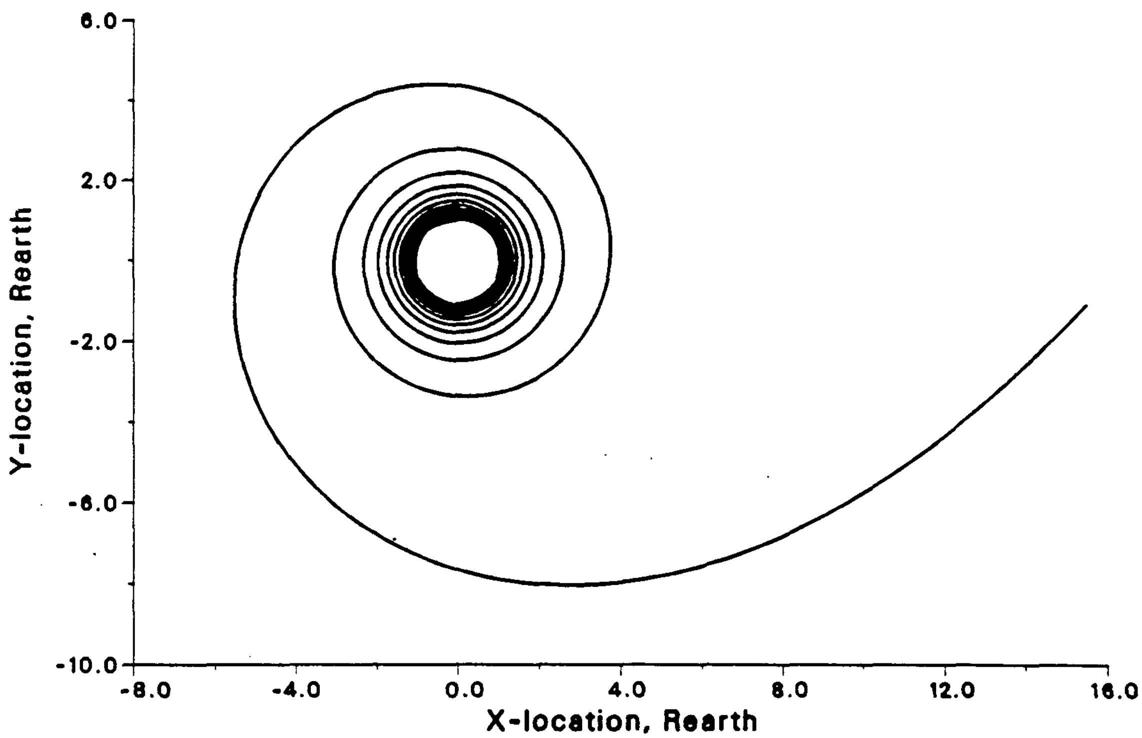

Figure 4.1: Maximum Energy Earth Escape Spiral for t_f =2.38 days

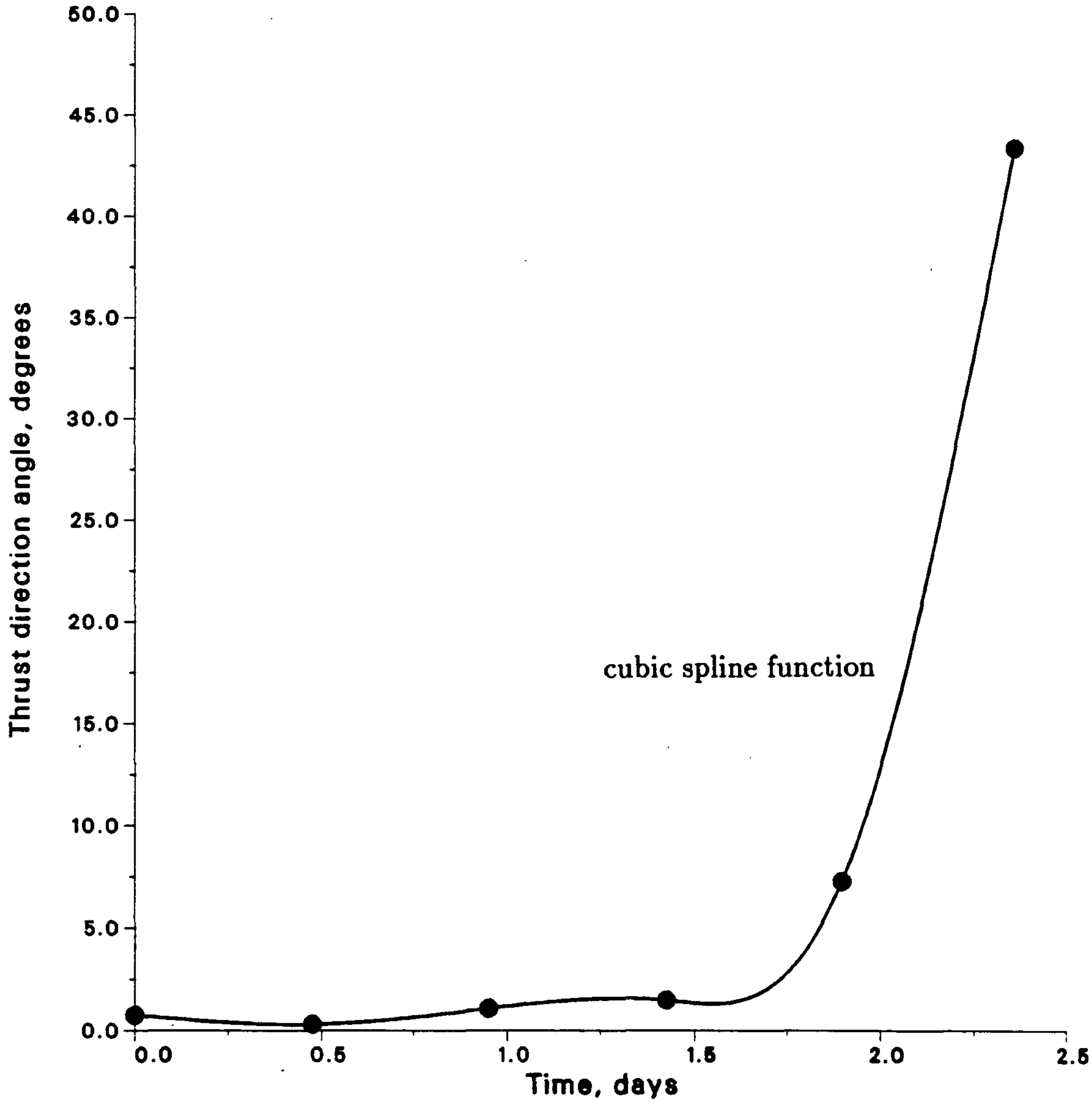

Figure 4.2: Optimal Thrust Direction Time History - Earth Escape Phase

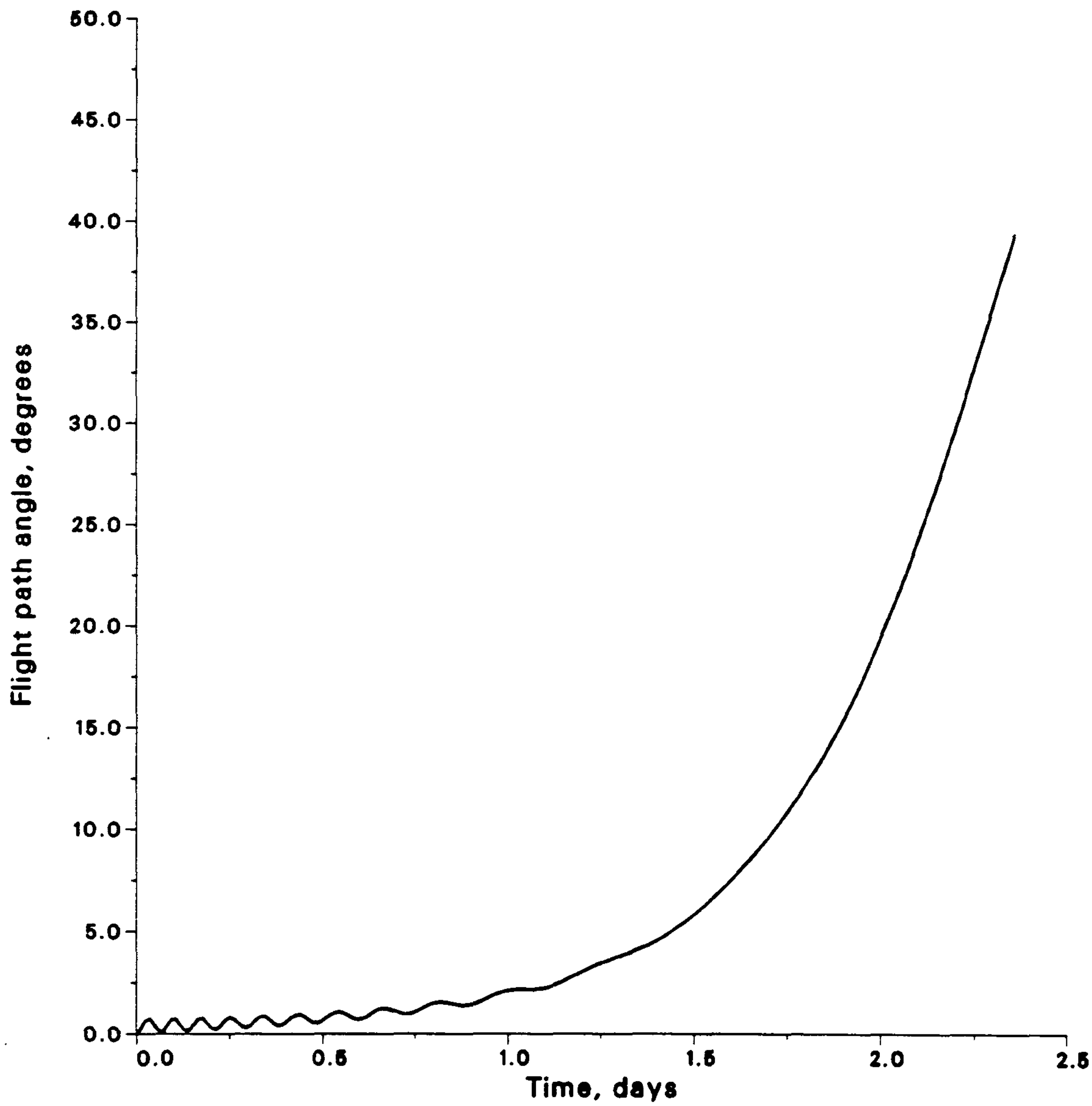

Figure 4.3: Optimal Flight Path Angle - Earth Escape Phase

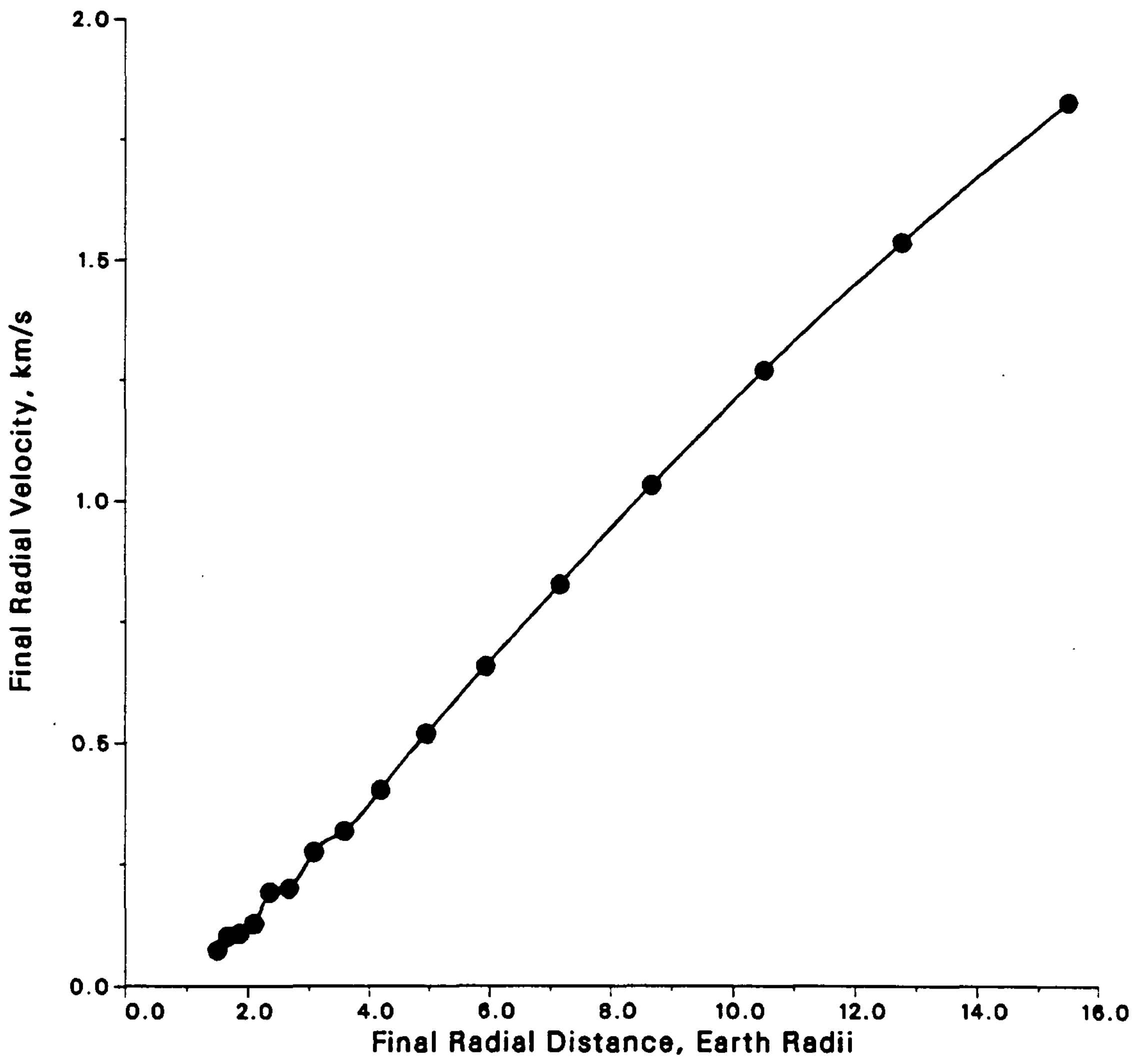

Figure 4.4: Final Radial Velocity vs Final Earth Radius

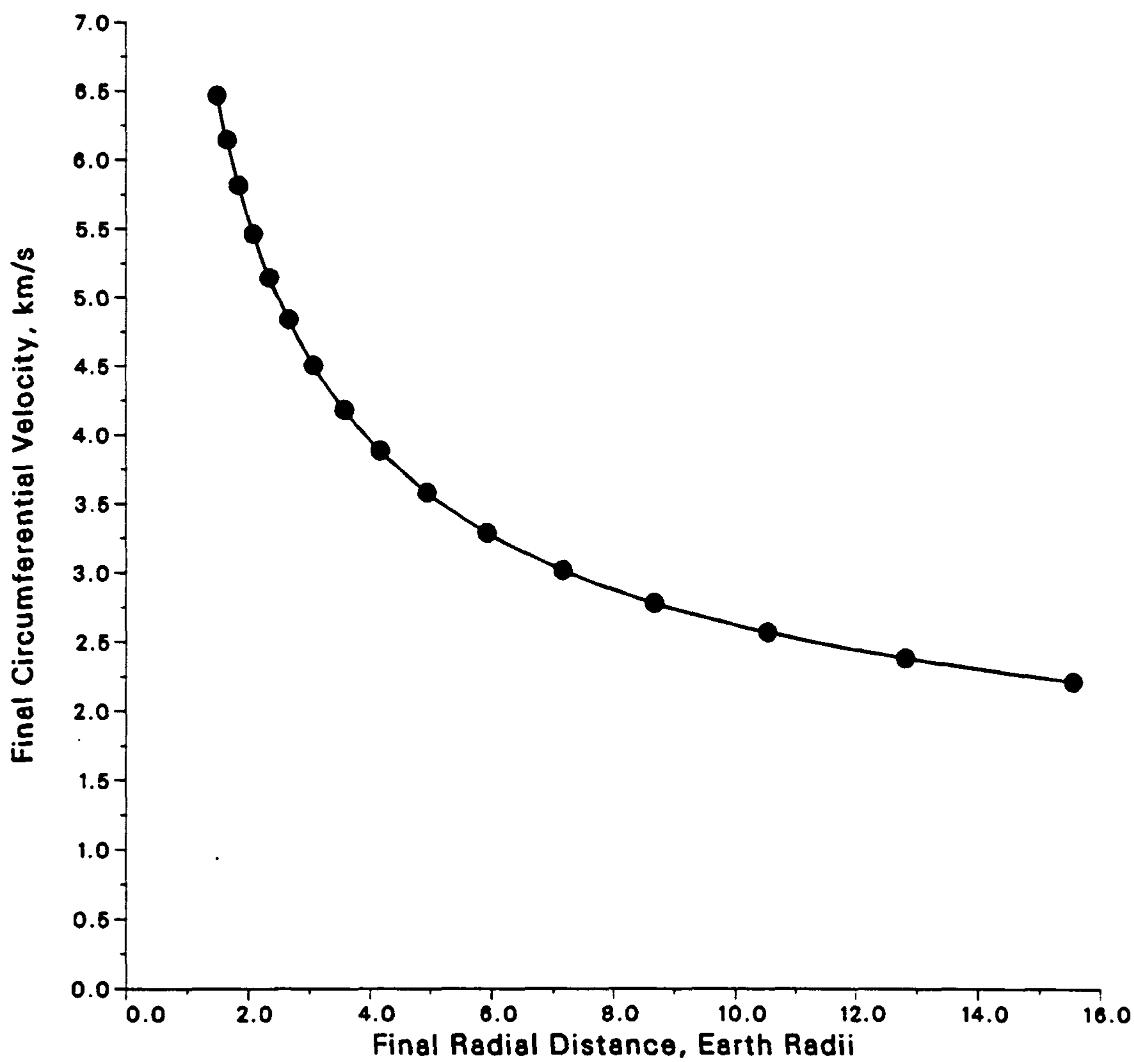

Figure 4.5: Final Circumferential Velocity vs Final Earth Radius

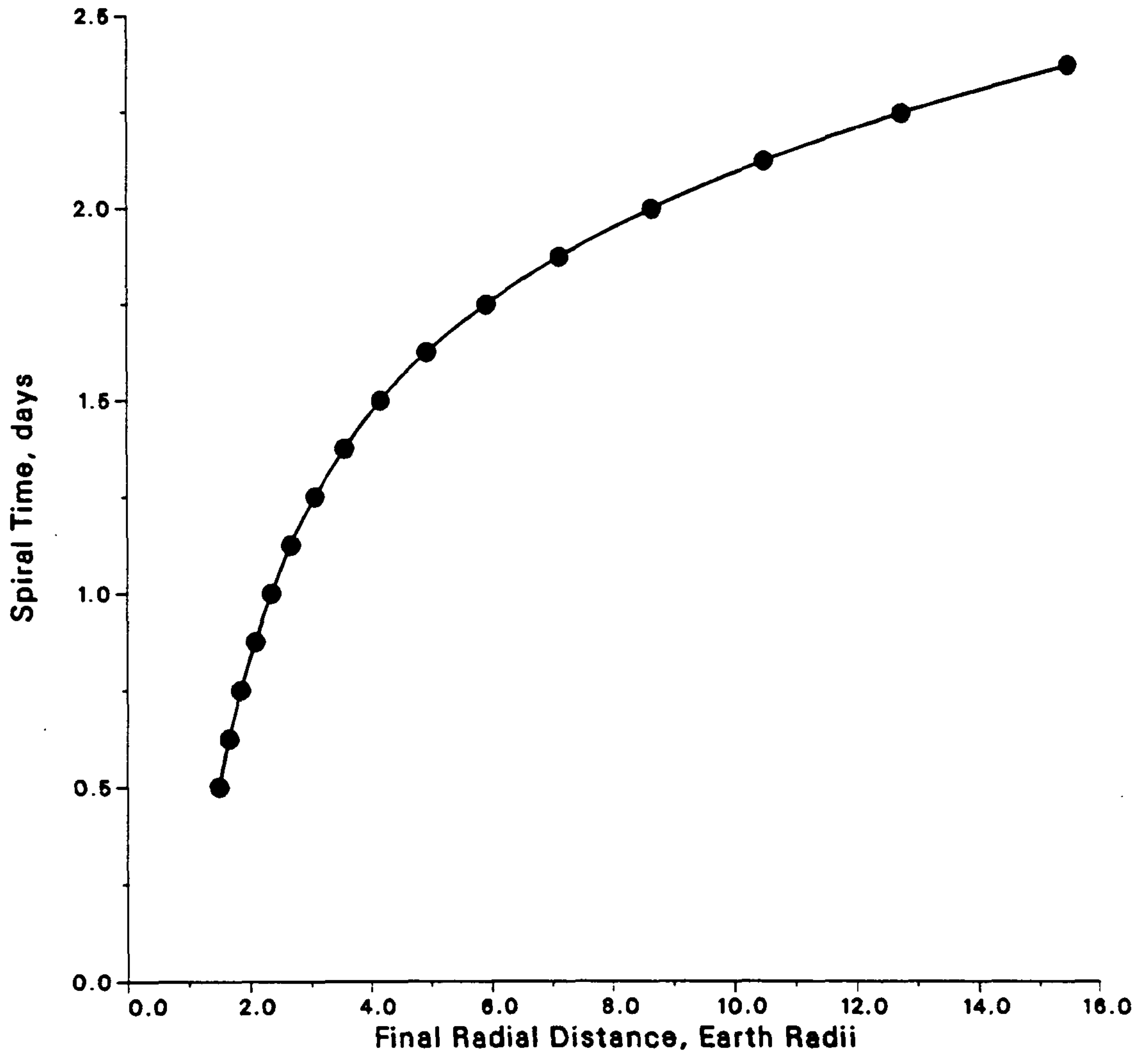

Figure 4.6: Escape Spiral Time vs Final Earth Radius

Optimal Moon Capture Trajectories

A range of fixed end-time maximum terminal energy Moon capture trajectories are solved in the same fashion as the Earth escape trajectories previously described. The objective is to provide terminal velocity vector boundary conditions in the vicinity of the Moon for the sub-optimal translunar phase. The optimal Moon capture trajectory is computed by integrating backwards in time starting from the circular 100 km Moon parking orbit and terminating at the maximum energy state in the prescribed flight time. The problem is to find the thrust direction time history parameterization for a continuous thrusting trajectory which maximizes energy at the end of a fixed time of flight. Two-body dynamics are implemented with the Moon as the central attracting body.

Since the capture trajectory is integrated backwards in time, the mass of the spacecraft increases as it spirals away from the Moon. Therefore, the mass of the spacecraft in the Moon parking orbit must be estimated as an initial condition for the backwards "capture" trajectory. A set of maximum energy trajectories are solved for a range of initial spacecraft masses in the Moon parking orbit.

A cubic spline function is fit through eleven control points to parameterize the thrust direction time history. The optimal thrust direction time history for the Moon capture phase is observed to be more oscillatory than the optimal thrust direction history for the Earth escape phase. The terminal energy for the Moon capture phase is also observed to be more sensitive to the number of control points than the Earth escape phase. These differences are attributed to the increased relative strength of the low-thrust engine in the weaker gravity field of the Moon. For these reasons, the number of control points is increased from six to eleven for the maximum energy

Moon capture trajectory.

The main objective of the maximum energy Moon capture trajectories is to provide a two-dimensional array of velocity vector and capture time data to be curve-fitted and used as the terminal boundary conditions near the Moon for the sub-optimal translunar phase. The two independent variables for the data array are the final radius from the Moon and spacecraft mass in the Moon parking orbit. In order to achieve the desired final radius, the proper corresponding flight time must be used in the fixed end-time maximum energy problem. This is accomplished by solving a sequence of maximum energy problems and adjusting the end-time. The end-time adjustment is computed by using the difference between the resulting radius and desired radius and the radial velocity at the final time. The end-time is adjusted and the next maximum energy trajectory is solved using the thrust direction control parameters from the previous solution as the initial guess. The sequence of maximum energy problems continues until the end-time adjustment becomes less than 0.02 *sec*.

Twenty maximum energy Moon capture trajectories are solved for five final radius values and four initial spacecraft masses. The longest capture trajectory spiralled for over 15.6 hours and terminated at 15 Moon radii which is less than half the distance to the Moon's Sphere of Influence. This is well within the accuracy of the two-body equations of motion since the Sphere of Influence is traditionally considered to be the boundary for two-body dynamics with respect to the Moon [9]. The lower limit of the estimated spacecraft mass in the circular Moon orbit is 86% of the initial total spacecraft mass. This is a conservative estimate based on London's [4] fuel requirements for an Earth-Moon trajectory utilizing the same thrust-to-weight ratio.

The optimal capture spiral trajectory for the heaviest spacecraft mass estimate and a fixed flight time of 15.6 hours days is presented in the Cartesian frame as shown in Figure 4.7. This maximum energy trajectory exceeds local Moon escape conditions and completes about 2.2 revolutions about the Moon. The optimal thrust direction time history is presented in Figure 4.8 and the corresponding flight path angle time history is shown in Figure 4.9. The parameterization of the thrust direction time history is shown by the cubic spline function fit through the eleven optimal control points. Similar to the optimal Earth escape trajectory, the thrust direction angle and flight path angle time histories show the same general shape.

The final velocity components and spiral times for the twenty trials are plotted against the final trajectory radii and initial spacecraft mass as shown in Figures 4.10, 4.11, and 4.12. The final circumferential velocities for the range of final Moon radial distances, as shown in Figure 4.11, are insensitive to the different initial spacecraft masses. These curves are used to supply the sub-optimal translunar phase with tractable boundary conditions.

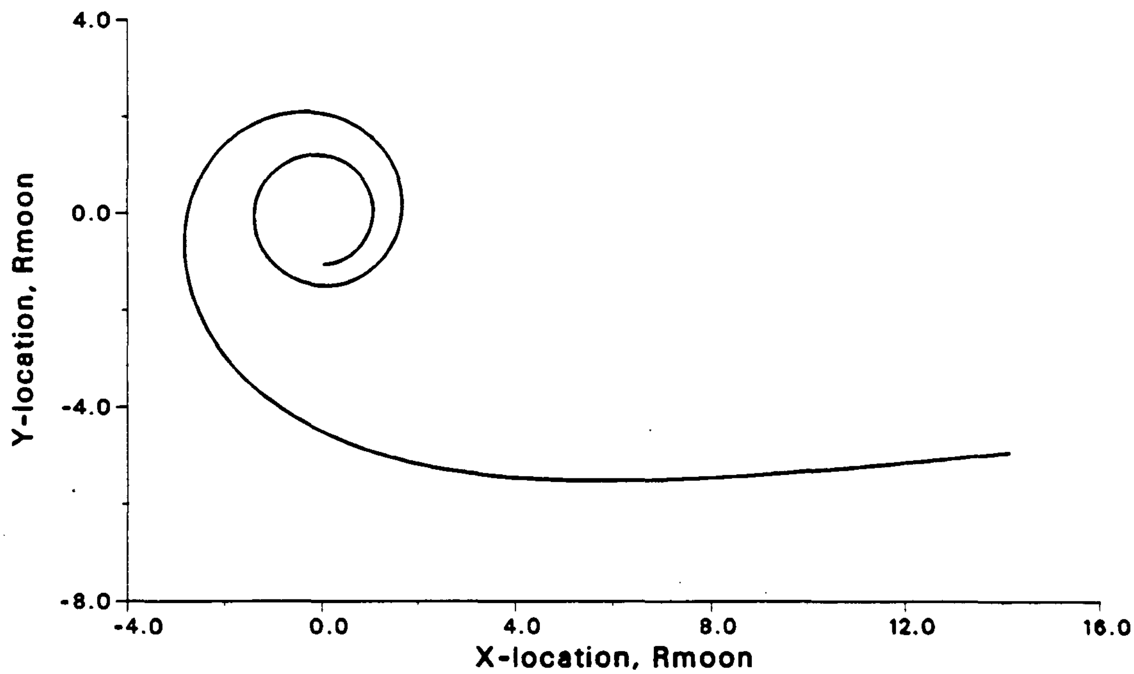

Figure 4.7: Maximum Energy Moon Capture Spiral for t_f =15.6 hours

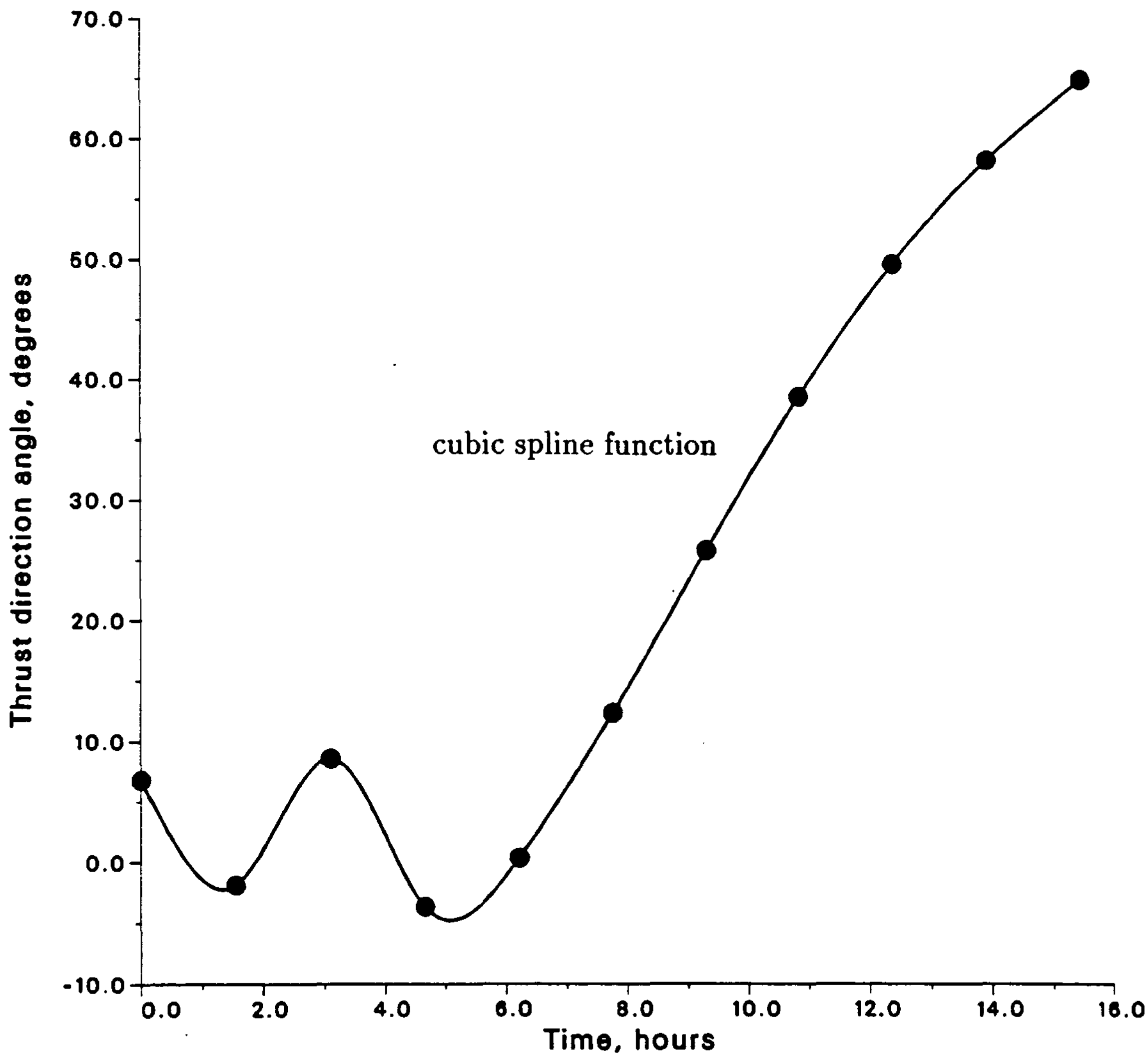

Figure 4.8: Optimal Thrust Direction Time History vs Final Moon Radius

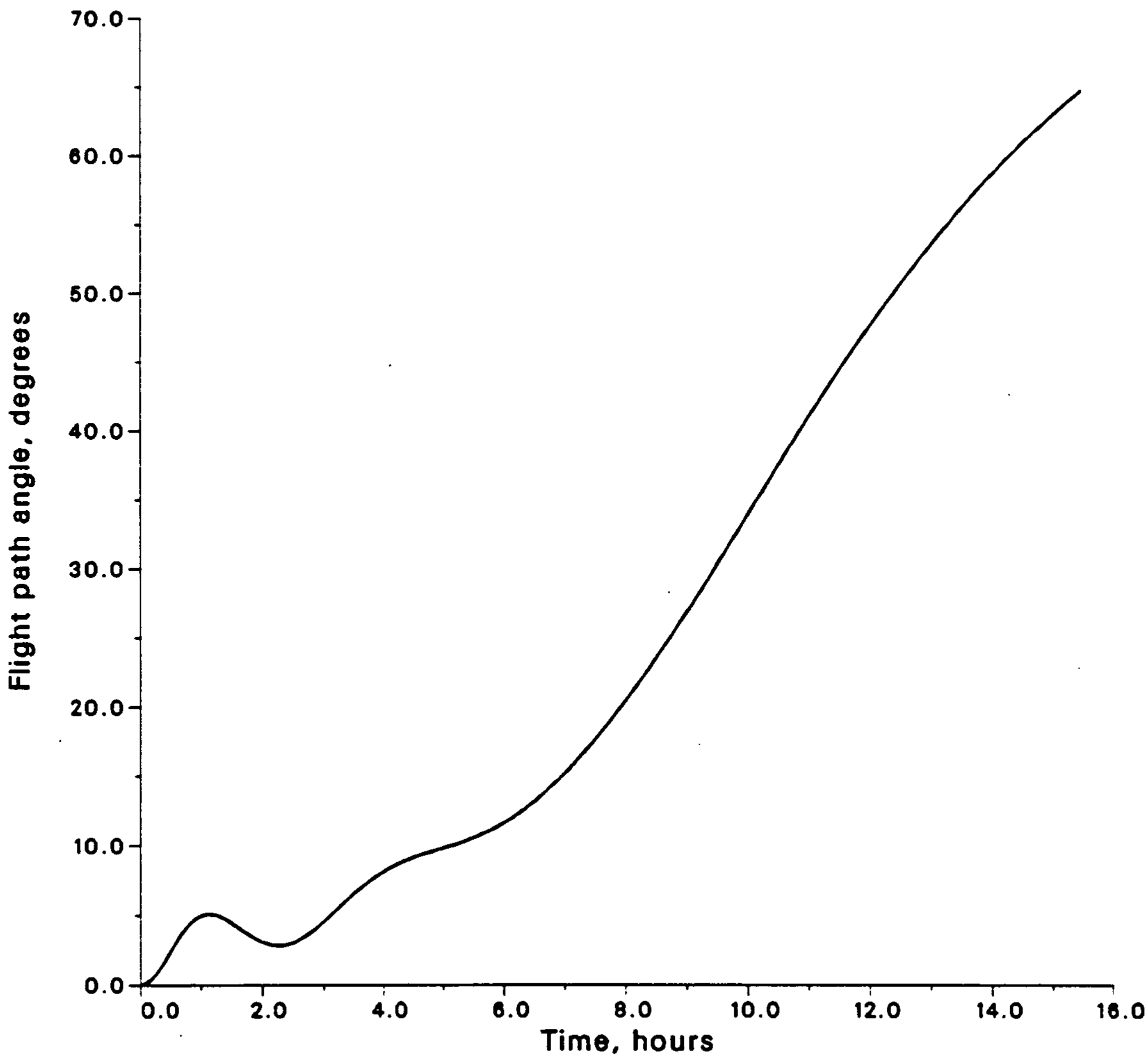

Figure 4.9: Optimal Flight Path Angle vs Final Moon Radius

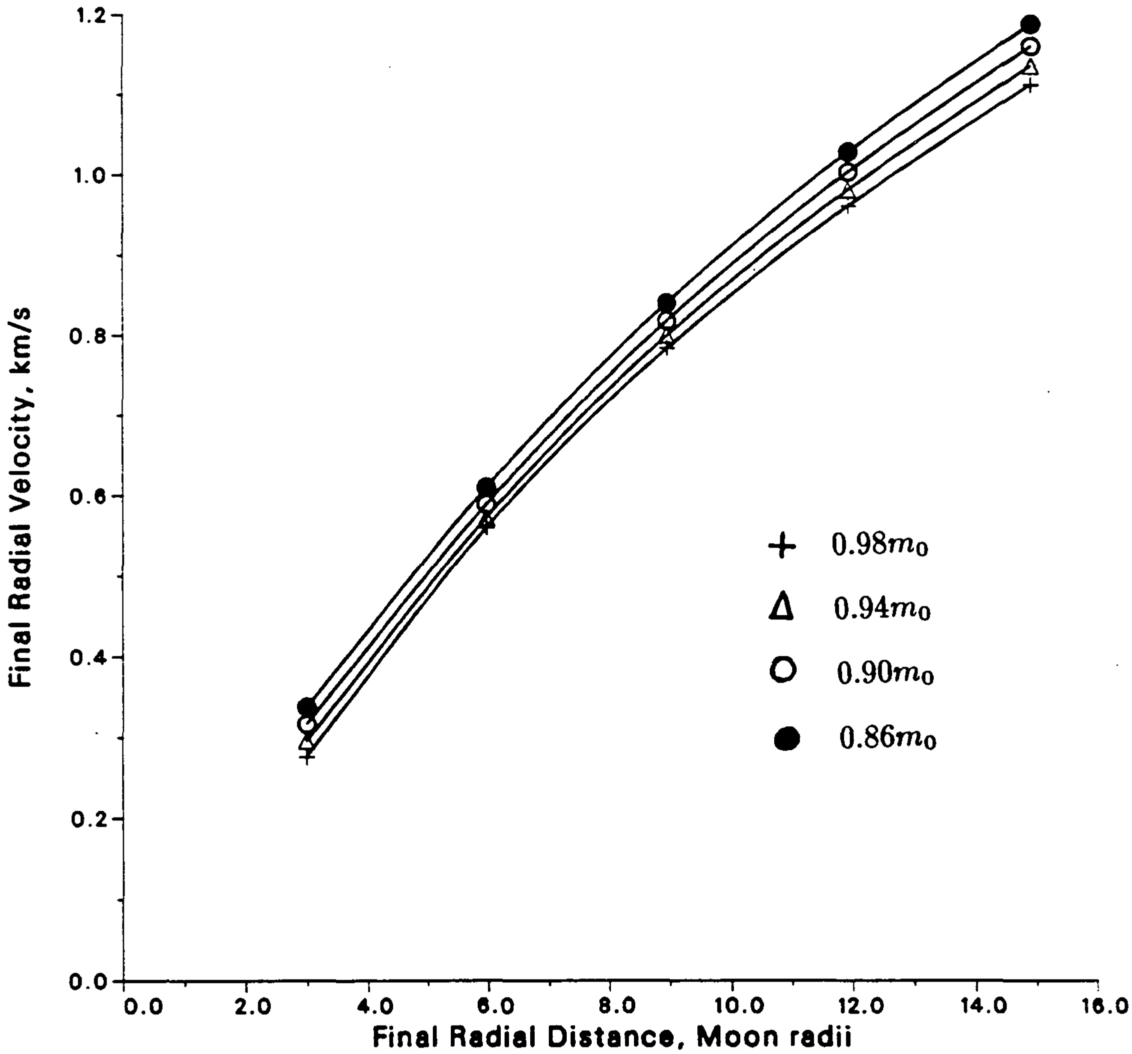

Figure 4.10: Final Radial Velocity Data vs Final Moon Radius

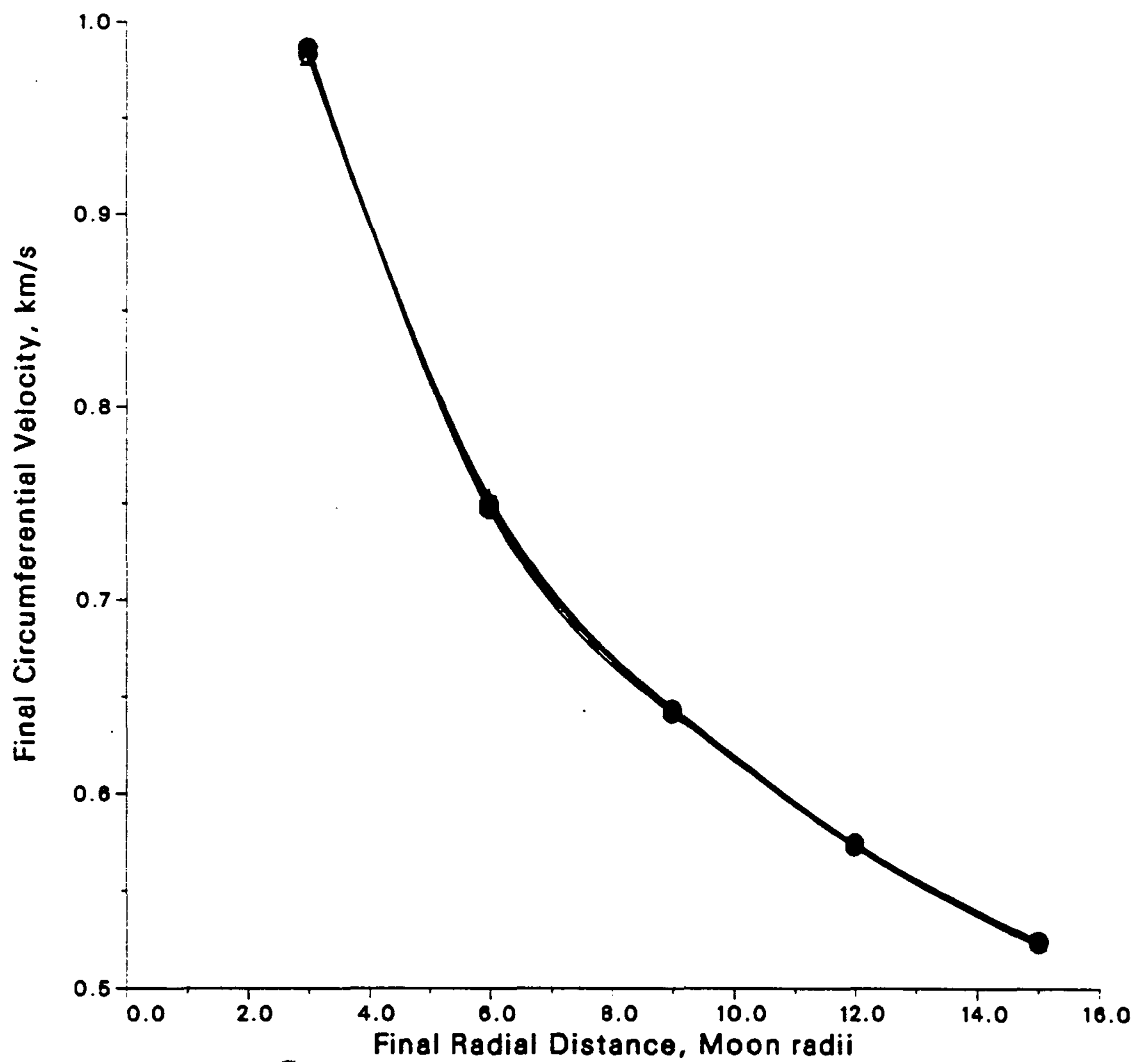

Figure 4.11: Final Circumferential Velocity vs Final Moon Radius

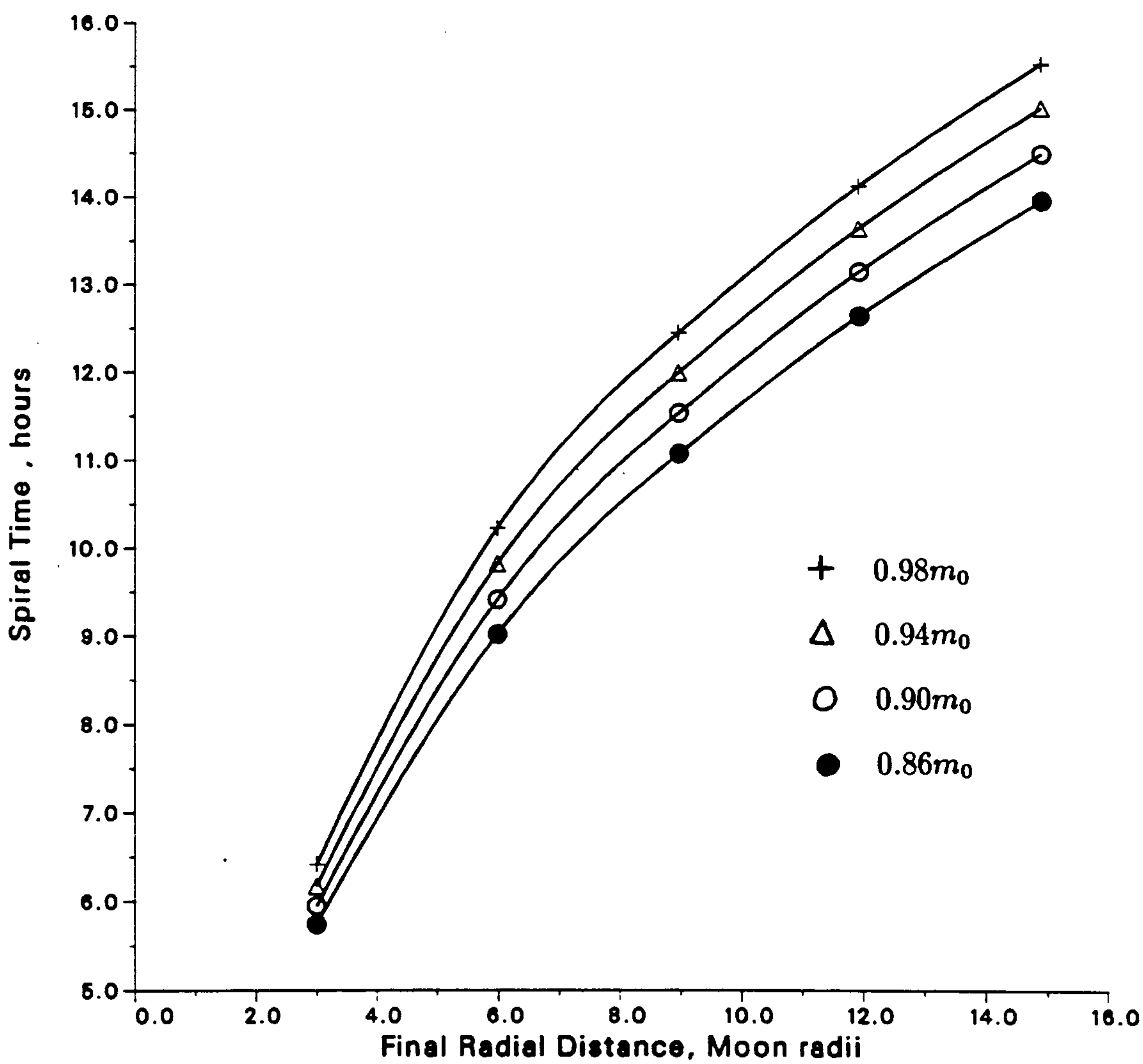

Figure 4.12: Capture Spiral Time

CHAPTER 5. SUB-OPTIMAL TRANSLUNAR TRAJECTORIES

The solutions to the maximum energy Earth escape and Moon capture trajectories from the previous chapter are used as boundary conditions for the sub-optimal translunar trajectory. The sub-optimal trajectories provide computationally inexpensive minimum fuel Earth-Moon trajectories and are essential to the solution of the optimal Earth-Moon trajectory. This chapter presents sub-optimal translunar trajectory solutions.

Problem Statement

The problem statement for the sub-optimal translunar trajectory problem using curve-fitted boundary conditions is as follows:

Minimize :

$$J = t_{escape} \ + \ t_{capture} \tag{5.1}$$

subject to

$$\dot{x}_1 = v_r = x_2 \tag{5.2}$$

$$\dot{x}_2 = \dot{v}_r = \frac{x_3^2}{x_1} + \tilde{a}_r \tag{5.3}$$

$$\dot{x}_3 = \dot{v}_\theta = -\frac{x_2 x_3}{x_1} + \tilde{a}_\theta \tag{5.4}$$

$$\dot{x}_4 = \dot{\theta} = \frac{x_3}{x_1} \tag{5.5}$$

and the boundary conditions

$$x_1(t_0) = r_1(t_0) \tag{5.6}$$

$$x_2(t_0) = f_1(\ r_1(t_0)\ ) \tag{5.7}$$

$$x_3(t_0) = f_2(\ r_1(t_0)\ ) \tag{5.8}$$

$$x_4(t_0) = \theta(t_0) \tag{5.9}$$

$$x_2(t_f) = g_1(\ m(t_f)\ ,\ r_2(t_f)\ ) \tag{5.10}$$

$$x_3(t_f) = g_2(\ m(t_f)\ ,\ r_2(t_f)\ ) \tag{5.11}$$

$$m_0 - \dot{m}(t_{escape} + t_{capture}) = m(t_f) \tag{5.12}$$

$$t_f = \alpha t_{nominal} \tag{5.13}$$

The objective is to find the coasting trajectory starting at the end of the Earth escape spiral and terminating at the initiation of the Moon capture spiral such that the sum of the total engine-on time for the two powered spiral phases is minimized. Since the propellant flow rate is constant, minimum total engine-on time corresponds to the minimum fuel trajectory. The performance index J consists of the time required for the maximum energy Earth escape spiral, t_{escape}, plus the maximum energy Moon capture spiral, $t_{capture}$. The translunar trajectory solution is termed sub-optimal since the two thrusting segments are replaced by the curve-fit results from two separate auxiliary problems, namely the maximum energy Earth escape and Moon capture trajectories. Since there is no guarantee that the maximum energy trajectories correspond to the minimum fuel Earth-Moon trajectory, the solution to the problem posed at this point is sub-optimal.

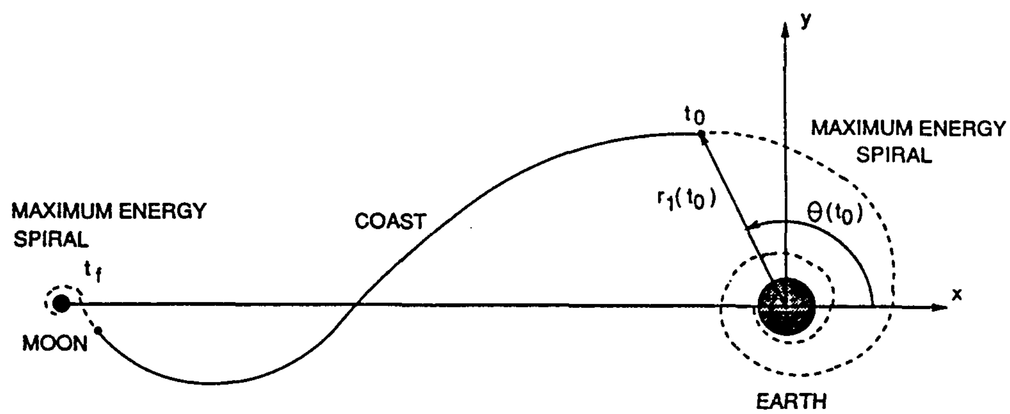

Figure 5.1: Schematic of the Sub-Optimal Translunar Trajectory

A schematic diagram of the trajectory is presented in Figure 5.1. The motion of the spacecraft is governed by the restricted three-body problem dynamics without the thrust terms, and the equations of motion are numerically integrated in a rotating, Earth-centered, polar coordinate frame. The restricted three-body problem dynamics require 500 integration steps for accurate gradient information since the gradients are computed with a first-order finite difference method.

The trajectory design vector consists of four elements: the initial Earth radial distance, $r_1(t_0)$, the initial Earth polar angle, $\theta(t_0)$, the final spacecraft mass in the Moon parking orbit, $m(t_f)$, and a time transformation variable, α. The first two control parameters are shown in Figure 5.1. The time transformation variable scales a fixed nominal end-time to create a free-end time problem as demonstrated by equation (5.13). The initial Earth polar angle defines the angular position of the spacecraft at the termination of the Earth escape spiral and the start of the translunar coast. The initial Earth radius and the final spacecraft mass are used to curve-fit the maximum energy Earth escape and Moon capture data, respectively.

Application of the Curve-Fits

The results of the maximum energy Earth escape trajectories are curve-fitted with the Earth radial distance as the independent variable. The sixteen trajectory points are fit using a variable-degree polynomial [15] with specified allowable errors for each curve. The curve-fitting routine starts with a second-order polynomial and increases the degree up to sixth-order until the specified errors are satisfied. The curve-fitted data typically requires a third or fourth-order polynomial. The curve-fit relations are

$$x_2(t_0) = v_r(t_0) = f_1(\ r_1(t_0)\ ) \tag{5.14}$$

$$x_3(t_0) = v_\theta(t_0) = f_2(\ r_1(t_0)\ ) \tag{5.15}$$

$$t_{escape} = f_3(\ r_1(t_0)\ ) \tag{5.16}$$

The initial radial and circumferential velocity components, $v_r(t_0)$ and $v_\theta(t_0)$, are computed by curve-fitting the maximum energy escape trajectory data with the design variable initial Earth radius, $r_1(t_0)$, as the independent variable. The curve-fitted radial and circumferential velocity components along with the control parameters $r_1(t_0)$ and $\theta(t_0)$ complete the initial state vector for the translunar phase. The Earth escape spiral flight time, t_{escape}, is also curve-fitted as a function of $r_1(t_0)$ and contributes to the performance index, total engine-on time.

The velocity components and capture spiral times from the maximum energy

Moon capture trajectories are functions of the Moon radial distance and the spacecraft mass in the Moon parking orbit. Therefore, a two-dimensional cubic spline function is used to curve-fit the velocity components. The control parameter $\dot{m}(t_f)$ and the final Moon radius at the end of the numerical integration, $r_2(t_f)$, are the respective independent variables. The Moon capture curve-fit relations are

$$x_2(t_f) = v_r(t_f) = g_1(\ m(t_f)\ ,\ r_2(t_f)\ ) \tag{5.17}$$

$$x_3(t_f) = v_\theta(t_f) = g_2(\ m(t_f)\ ,\ r_2(t_f)\ ) \tag{5.18}$$

$$t_{capture} = g_3(\ m(t_f)\ ,\ r_2(t_f)\ ) \tag{5.19}$$

The curve-fitted radial and circumferential velocity components are two terminal state constraints that must be equal to the resulting velocity components at the end of the numerical integration of the equations of motion. The absolute value of the resulting circumferential velocity component is compared with the curve-fitted velocity component therefore allowing either a posigrade or retrograde parking orbit about the Moon. The curve-fitted capture spiral time contributes to the performance index and is also used to compute the final mass of the spacecraft in the circular Moon parking orbit. The computed final spacecraft mass, as shown by equation (5.12), is the third equality constraint and must be equal to the estimated final mass used in the curve-fitting, control parameter $m(t_f)$.

Numerical Results

The numerical simulation of the coasting trajectory from the end of the Earth escape spiral to the beginning of the Moon capture spiral is very sensitive to the initial guess. Before the sequential quadratic programming optimization method is used on the translunar problem, an adequate guess of the four control parameters is required. The initial estimate of the control parameters must result in a trajectory that terminates near the Moon and is still approaching the Moon. A series of non-optimal trajectories are computed with different control vector estimates until the spacecraft is observed to terminate within the Moon's Sphere of Influence with negative radial velocity.

Table 5.1: Initial Control Parameter Estimates and Characteristics of the Initial Trajectory

Trial	$r_1(t_0)$, Re	$\theta(t_0)$, rad	$m(t_f)$, N.D.	α, N.D.	$r_2(t_f)_{i=1}$
1	12.50	2.50	0.94	1.00	9.64
2	12.53	2.50	0.94	1.00	5.24
3	12.50	2.53	0.94	1.00	15.80
4	12.50	2.50	0.86	1.00	9.64
5	12.50	2.50	0.94	1.04	4.76

Several sub-optimal translunar trajectories are solved with different initial estimates of the control parameters. The termination tolerances are set at 10^{-4} for the change in performance index between iterations and the norm of the Lagrangian gradient. The performance index, total engine-on time, is measured in dimensionless time and the termination tolerance 10^{-4} corresponds to a change of 38 seconds between iterations. The tolerance for the equality constraints is 10^{-8}. The velocity equality constraints are expressed in km/sec while the final mass constraint is expressed in percentage of the initial mass.

The range of initial control parameter estimates and characteristics of the initial trajectory are shown in Table 5.1. The variable $r_2(t_f)_{i=1}$ is the distance from the Moon to the spacecraft in Moon radii at the final end-time for the first iteration.

The small changes in the initial control parameters produce substantial changes in the initial trajectory. The different initial control estimates produce initial trajectories that terminate at very close range and at great distance to the Moon. Trial 1, the baseline solution, has an initial control estimate that produces a coasting trajectory which terminates at 9.64 Moon radii for the first iteration. The initial Earth radius estimate, $r_1(t_0)$, is perturbed by 0.2% for Trial 2 and results in an initial trajectory which terminates at 5.24 Moon radii. Trial 3 has an initial polar angle perturbed by 1.72 degrees from the baseline and the first iteration trajectory terminates at 15.80 Moon radii which is outside of the Moon capture curve-fit data. Trial 4 has a significantly different final mass estimate and results in essentially the same first iteration trajectory as the baseline estimate. Trial 5 has the estimated coast time perturbed by 4.1 hours and terminates at 4.76 Moon radii. All of the trials converge in 15-20 iterations to the same solution exhibiting a total engine-on time of 2.682 days which demonstrates the robustness of the problem. The solutions to the trials also show identical optimal control parameters.

The sub-optimal translunar trajectory begins at a radius of 12.50 Earth radii with an Earth polar angle of 145.3 degrees. The curve-fitted radial and circumferential Earth-relative velocities are 1.50 and 2.41 km/sec at the start of the translunar coast which corresponds to an Earth-relative energy level of -0.964 km^2/sec^2 and an eccentricity of 0.742. The spacecraft coasts for 4.57 days until the coasting translunar trajectory terminates at 7.29 Moon radii with a Moon polar angle of 348.5 degrees. The curve-fitted radial and circumferential Moon-relative velocities are -0.68 and 0.70 km/sec at the end of the translunar coast which corresponds to a Moon-relative energy level of 0.089 km^2/sec^2 and an eccentricity of 1.256. The resulting circular

Moon parking orbit is therefore posigrade since the circumferential velocity is positive. The sub-optimal trajectory requires a fuel to initial mass ratio of 6.92% and the resulting final spacecraft mass in the low Moon orbit is 93,081 kg. The sub-optimal trajectory solution is shown in Figure 5.2 in the Earth-centered, rotating coordinate system.

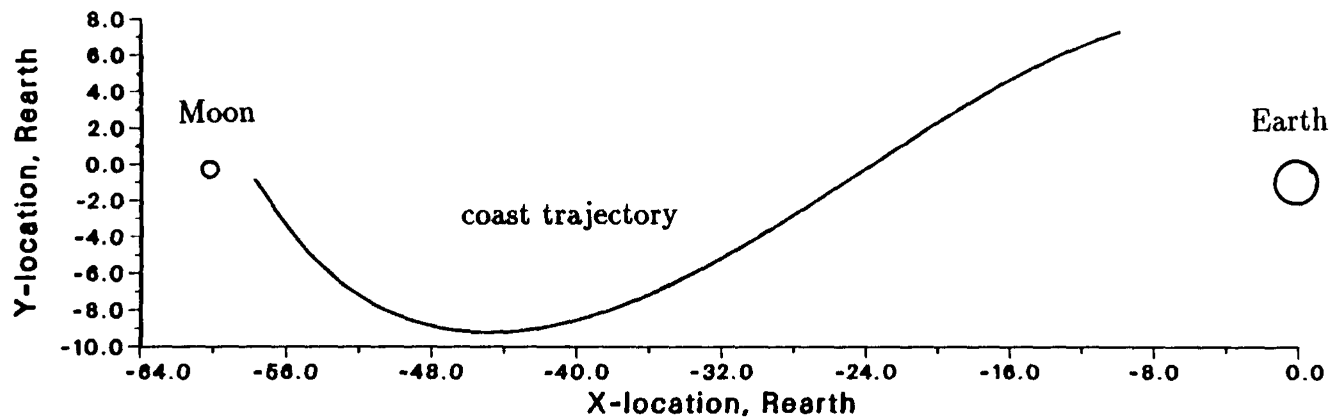

Figure 5.2: Sub-Optimal Translunar Trajectory

CHAPTER 6. OPTIMAL EARTH-MOON TRAJECTORIES

The minimum fuel Earth-Moon trajectory uses the sub-optimal trajectory solution as an initial estimate of the trajectory design parameters. This chapter presents the solution of the minimum fuel Earth-Moon trajectory and provides numerical results.

Problem Statement

The problem statement for the optimal Earth-Moon minimum fuel trajectory problem is as follows:

Minimize :

$$J = (t_{off} - t_0) \; + \; (t_f - t_{on}) \tag{6.1}$$

subject to

$$\dot{x}_1 = v_r = x_2 \tag{6.2}$$

$$\dot{x}_2 = \dot{v}_r = \frac{x_3^2}{x_1} + \tilde{a}_r \tag{6.3}$$

$$\dot{x}_3 = \dot{v}_\theta = -\frac{x_2 x_3}{x_1} + \tilde{a}_\theta \tag{6.4}$$

$$\dot{x}_4 = \dot{\theta} = \frac{x_3}{x_1} \tag{6.5}$$

and the boundary conditions

$$x_1(t_0) = r_1(t_0) = 315 \ km \quad LEO \tag{6.6}$$

$$x_2(t_0) = v_r(t_0) = 0 \tag{6.7}$$

$$x_3(t_0) = v_\theta(t_0) = \sqrt{\frac{\mu}{r_1(t_0)}} \tag{6.8}$$

$$x_4(t_0) = \theta(t_0) \tag{6.9}$$

$$x_1(t_f) = r_2(t_f) = 100 \ km \quad LLO \tag{6.10}$$

$$x_2(t_f) = v_r(t_f) = 0 \tag{6.11}$$

$$x_3(t_f) = v_\theta(t_f) = \sqrt{\frac{\mu}{r_2(t_f)}} \tag{6.12}$$

$$t_f = \alpha t_{nominal} \tag{6.13}$$

The objective is to find the total Earth-to-Moon trajectory starting at the circular Earth parking orbit and terminating at the circular Moon parking orbit such that the sum of the total engine-on time (and therefore fuel required) is minimized. The performance index expression, equation (6.1), can be rewritten:

$$J = t_{escape} + t_{capture} \tag{6.14}$$

where

$$t_{escape} = t_{off} - t_0 \tag{6.15}$$

$$t_{capture} = t_f - t_{on} \tag{6.16}$$

The powered spiral times for the Earth escape and Moon capture trajectories, t_{escape} and $t_{capture}$, are computed by using the engine-off and on switch times, t_{off}

and t_{on}. The switch time t_{off} defines the end of the continuous thrust Earth escape spiral while t_{on} defines the time for beginning the continuous thrust Moon capture spiral. The final end-time, t_f, is free and is computed by scaling a nominal end-time with the time transformation variable α as described by equation (6.13).

The motion of the spacecraft is governed by the restricted three-body problem dynamics and the equations of motion are numerically integrated in both Earth-centered and Moon-centered, rotating polar coordinate frames. The total simulation is composed of three separate phases: the continuous thrust Earth escape spiral, the translunar coast, and the continuous thrust Moon capture spiral. The Earth escape spiral and translunar coast trajectory are both numerically integrated in an Earth-centered rotating polar coordinate frame. The Moon capture spiral is integrated in a Moon-centered polar coordinate system for increased numerical accuracy. Each phase has a separate integration step size which is required for the accuracy of that phase. The two spiral phases require smaller integration step sizes than the translunar coast since the motion is cyclic. The total Earth-Moon simulation requires 2715 integration steps for an accurate trajectory and accurate gradient information.

The Earth-Moon trajectory must terminate in a circular, 100 km parking orbit around the Moon. Originally, only two equality constraints were imposed which required the final altitude of the Moon parking orbit be 100 km and the final eccentricity be zero. This, however, resulted in convergence problems since eccentricity is a single-valued function and can never be negative. The single-valued nature of eccentricity results in discontinuities in the respective gradients and thus causes convergence problems. For this reason, the eccentricity equality constraint is replaced by two terminal state constraints requiring that the final Moon-relative radial velocity be zero and the final circumferential velocity be equal to the circular orbital speed at the desired 100 km altitude. Again, the absolute value of the final circumferential velocity is equated to the circular orbital speed to allow both posigrade and retrograde parking orbits. The three terminal state constraints are shown by equations (6.10)- (6.12).

The trajectory design vector consists of twenty-one parameters: six control points for the Earth escape thrust direction time history, the Earth orbit departure polar angle, $\theta(t_0)$, the engine-off switch time, t_{off}, the engine-on switch time, t_{on}, eleven control points for the Moon capture thrust direction time history, and the time transformation variable, α. The Earth orbit departure polar angle, $\theta(t_0)$, defines the angular position of the spacecraft with respect to the Earth-Moon line when the escape spiral is initiated and the spacecraft departs the Earth parking orbit.

Applying the Sub-Optimal Solution

The solution of the sub-optimal translunar trajectory problem is used to provide an initial estimate of the twenty-one element control parameter vector. The Earth escape and Moon capture spiral times are computed in the sub-optimal problem by curve-fitting the corresponding maximum energy spiral trajectories. These spiral times, along with the translunar coast time for the sub-optimal solution, provide estimates for the switch times t_{on} and t_{off} and the time transformation parameter α. Two auxiliary maximum energy Earth escape and Moon capture trajectories are solved for the respective spiral times from the sub-optimal solution. The resulting optimal thrust direction parameterizations provide estimates for the six Earth escape and eleven Moon capture control points. Finally, the estimated Earth orbit departure polar angle, $\theta(t_0)$, is computed by subtracting the number of revolutions for the maximum energy escape spiral from the Earth polar angle that defines the initiation of the translunar coast for the sub-optimal solution:

$$\theta(t_0) = \theta(t_0)_{sub} - 2\pi k \tag{6.17}$$

The parameter $\theta(t_0)_{sub}$ is the polar angle from the sub-optimal solution which defines the initiation of the translunar coast and k is the number of revolutions around the Earth during the maximum energy trajectory.

This initial control parameter estimate, as previously mentioned, is derived from the sub-optimal solution and the maximum energy trajectories and therefore does not include the restricted three-body problem dynamics during the continuous thrust spirals. The initial control estimate produces an Earth-Moon trajectory that terminates in the vicinity of the Moon at an altitude of 164 km and an eccentricity of 0.05. The Moon parking orbit resulting from the initial control parameter vector estimate has a 180 km apoapsis altitude and a periapsis which just skims below the surface of the Moon.

Numerical Results

The minimum fuel Earth-Moon trajectory is solved by using the sub-optimal solution to provide an initial estimate of the twenty-one control parameters as previously described.

The termination tolerances are set at 10^{-6} for the change in performance index between iterations and the norm of the Lagrangian gradient. The performance index, total engine-on time, is measured in dimensionless time and the tolerance 10^{-6}

corresponds to a change of 0.4 *sec* between iterations. The tolerance for the equality constraints is 10^{-8}. The problem converges to a solution after 21 iterations when the change in performance index between iterations becomes less than 10^{-6}.

The optimal Earth-Moon trajectory begins with a continuous thrust Earth escape spiral which reaches an Earth-relative energy level of $-0.963\ km^2/sec^2$ and an eccentricity of 0.742 after 2.23 days. The low-thrust engine is shut-off at this point when the spacecraft is at 12.49 Earth radii with a polar angle of 145.6 degrees. The spacecraft coasts for 4.63 days until it reaches a distance of 7.22 Moon radii with a Moon-relative energy of $0.084\ km^2/sec^2$ and an eccentricity of 1.241. The spacecraft then restarts the engine and follows a continuous thrust capture spiral for 10.7 hours and terminates on a posigrade, circular 100 *km* Moon parking orbit. The total engine-on time for the optimal Earth-Moon trajectory is 2.679 days and the total Earth-Moon transfer time is 7.31 days. The Earth-Moon trajectory requires a fuel to initial mass ratio of 6.91% and the resulting final spacecraft mass in the low Moon orbit is 93,088 *kg*. The minimum fuel trajectory is presented by Figures 6.1 and 6.2 in both rotating and non-rotating coordinate frames.

The optimal Earth-Moon trajectory is nearly identical to the sub-optimal translunar trajectory solution. The minimum fuel solution reduces the total engine-on time by about 4 minutes and increases the final spacecraft mass by about 7 *kg* in comparison with the sub-optimal solution. However, though very small changes in the control parameters and performance index are observed, the final Moon parking orbit was altered from a $0 \times 180\ km$ altitude elliptical orbit for the initial control parameter estimate to a 100 *km* circular orbit for the optimal trajectory. This demonstrates how little effect the low-thrust engine has on the Earth-Moon trajectory.

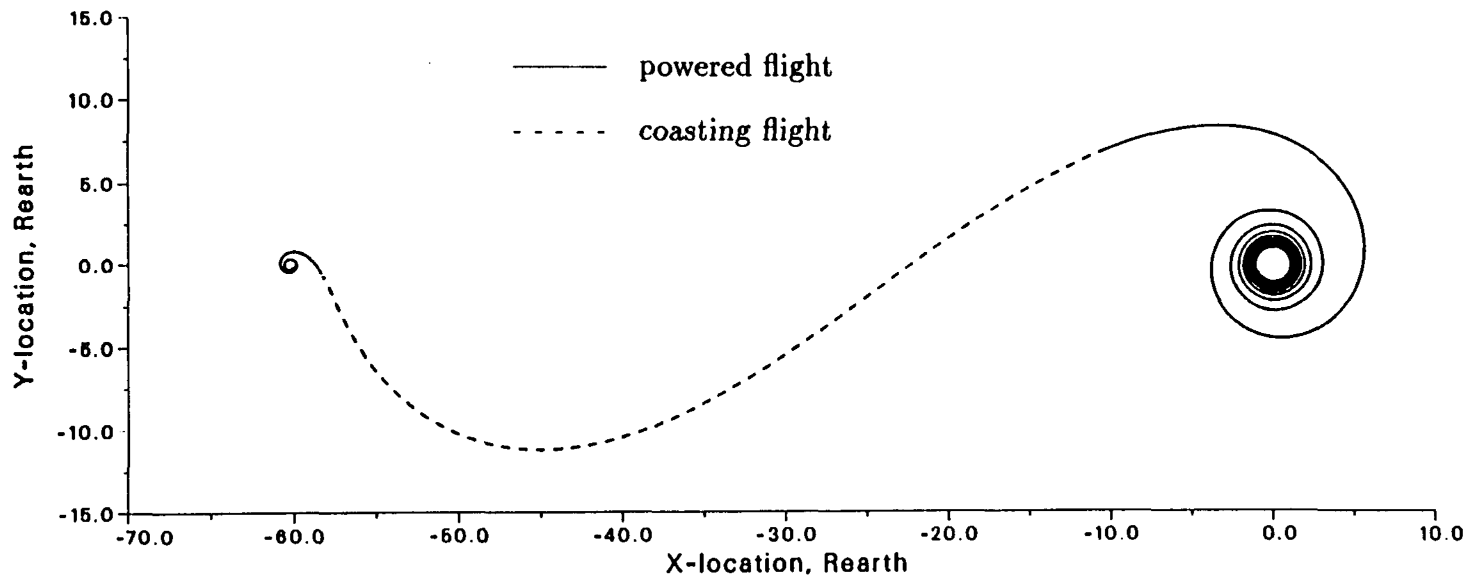

Figure 6.1: Optimal Earth-Moon Trajectory - rotating coordinates

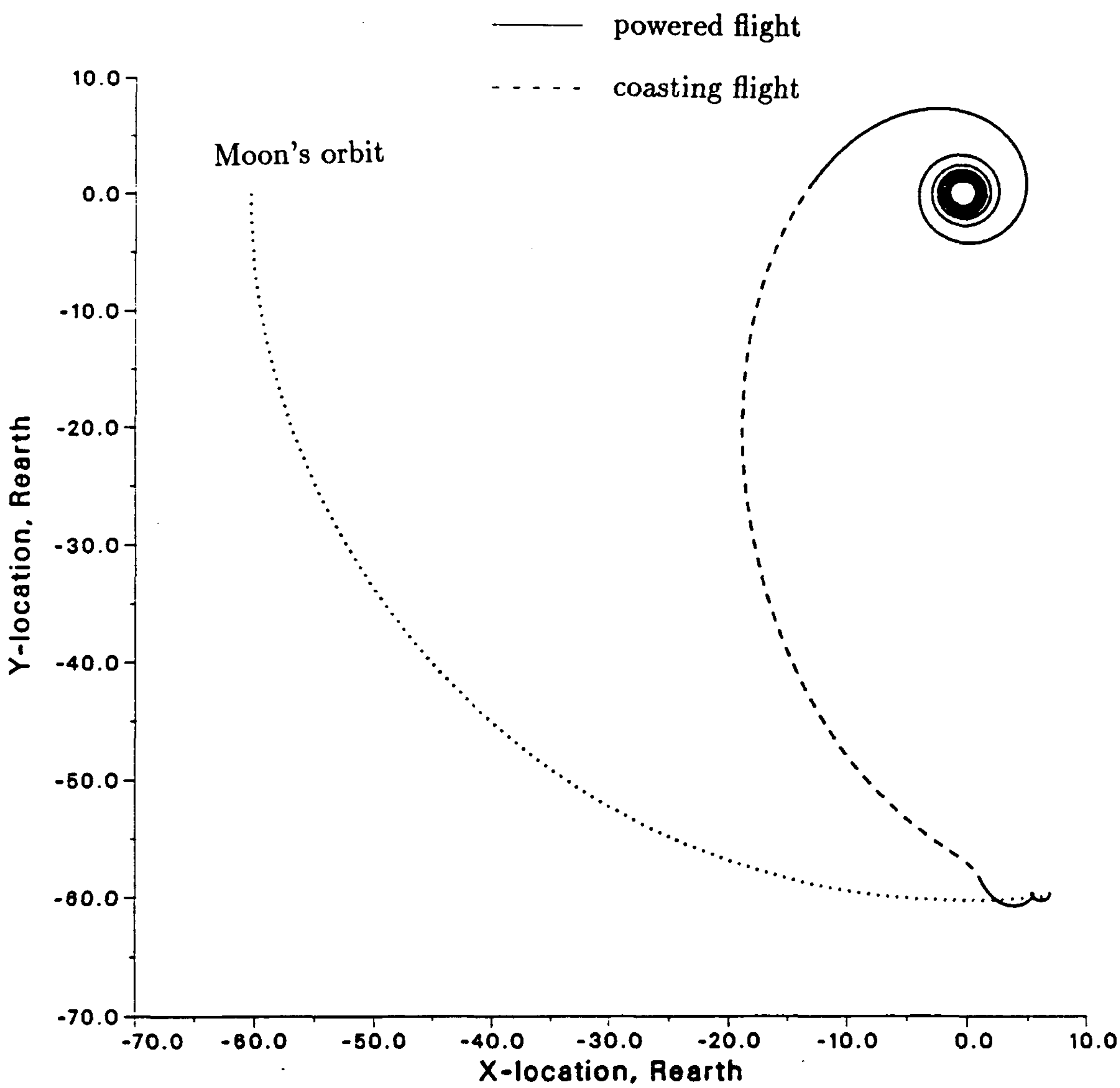

Figure 6.2: Optimal Earth-Moon Trajectory - non-rotating coordinates

CHAPTER 7. SUMMARY AND CONCLUSIONS

Minimum fuel, low thrust, Earth-Moon trajectories have been obtained using a direct optimization method. The infinite-dimension optimal control problem is solved numerically as a finite parameter optimization problem by a sequential quadratic programming method. The Earth-Moon trajectory is a planar transfer between circular Earth and Moon parking orbits, and the equations of motion are governed by the restricted three-body problem dynamics. The minimum fuel trajectory is ultimately obtained by solving a sequence of three sub-problems: maximum energy Earth escape and Moon capture trajectories, sub-optimal translunar trajectories, and finally the optimal Earth-Moon minimum fuel trajectory. The solution of each sub-problem provides essential data for boundary conditions and initial estimates of the control parameters for the next sub-problem.

This method of solution has proven to be an effective and systematic approach to solving the minimum fuel problem. The presented method can be applied in principle to solve minimum fuel Earth-Moon trajectories for any desired initial thrust-to-weight ratio. The optimal Earth-Moon trajectory problem, as presented in Chapter 6, is an extremely complex simulation with very sensitive equations of motion and stringent equality constraints. The problem also requires a large number of integration steps for accuracy and is therefore a computationally demanding problem. The Earth-Moon trajectory is extremely sensitive to the initial estimate of the control parameters. The control parameters essentially dictate the sequencing and programming of the two thrusting segments about the primary bodies. A very good initial control estimate is obtained by the solution of the sub-optimal translunar trajectory. The sub-optimal problem solves the minimum fuel coasting trajectory between boundary conditions obtained by curve-fitting spiral trajectory data about the Earth and Moon. The sub-optimal problem is robust and much easier to solve than the optimal Earth-Moon trajectory since the simulation is much less complex than the total Earth-Moon simulation. The sub-optimal trajectory provides a very good solution since the optimal Earth-Moon trajectory provides only a 0.008% additional savings in fuel requirements at a much higher computational cost. The optimal Earth-Moon trajectory is nearly identical to the sub-optimal solution. The Earth parking orbit-to-Moon parking orbit solution, however, includes three-body dynamics throughout the trajectory and shows significant alterations in the initial trajectory estimate without any significant changes in the performance index. This demonstrates how little effect the control parameters and therefore the low-thrust engine has on the Earth-Moon trajectory.

Several extensions to the minimum fuel, low-thrust, Earth-Moon trajectory problem exist. Three-dimensional, non-planar transfers between Earth and Moon parking orbits can be researched. Lower initial thrust-to-weight ratios can also be investigated. The curve-fitting procedure can possibly be replaced by analytical functions for the continuous thrust Earth escape and Moon capture spirals. The problem can also include the optimization of engine parameters such as specific impulse and input power.

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APPENDIX A. THREE-BODY EQUATIONS OF MOTION IN A MOON-CENTERED, ROTATING FRAME

The equations of motion of the spacecraft referenced to a Moon-centered, rotating polar coordinate system are derived in this appendix. The dynamics are from the restricted three-body problem. The absolute acceleration of the spacecraft in a Moon-centered, inertial (non-rotating) Cartesian frame is analogous to the Earth-centered, inertial acceleration as presented by Egorov [11]:

$$\ddot{X} = -\frac{Gm_2X}{R_2^3} + \frac{Gm_1(X_E - X)}{R_1^3} - \frac{Gm_1X_E}{D^3} \tag{A.1}$$

$$\ddot{Y} = -\frac{Gm_2Y}{R_2^3} + \frac{Gm_1(Y_E - Y)}{R_1^3} - \frac{Gm_1Y_E}{D^3} \tag{A.2}$$

where

X : x-location of the spacecraft (relative to Moon)
Y : y-location of the spacecraft (relative to Moon)
X_E : x-location of the Earth (relative to Moon)
Y_E : y-location of the Earth (relative to Moon)
R_1 : Earth to spacecraft distance
R_2 : Moon to spacecraft distance
D : Earth to Moon distance

These equations represent the absolute acceleration of a non-thrusting spacecraft influenced only by the gravitational forces of the two primaries in an Moon-centered, inertial frame. The above equations are of the same form as the Earth-centered absolute accelerations shown in equations (2.15) and (2.16). The coordinate system is now fixed at the center of the Moon and the position of the Earth relative to the Moon is required. The third term in each equation is the centripetal acceleration.

The thrust acceleration components in the Moon-centered Cartesian frame are:

$$a_{T_x} = a_T \sin u \cos \theta_M - a_T \cos u \sin \theta_M \tag{A.3}$$

$$a_{T_y} = a_T \sin u \sin \theta_M + a_T \cos u \cos \theta_M \tag{A.4}$$

Again, a_T is the thrust acceleration magnitude computed by the linear relation during the numerical integration process:

$$a_T(t) = \frac{T}{m_0 - \dot{m}t} \tag{A.5}$$

The thrust direction angle, u, is measured with respect to the local Moon horizon. The Moon-centered polar angle, θ_M, is measured from the positive x-axis to the radius vector from the Moon to the spacecraft. The thrust acceleration components are added to the gravitational acceleration components from equations (A.1) and (A.2) to form the absolute acceleration of the thrusting spacecraft in the Moon-centered, inertial frame:

$$\ddot{X} = -\frac{Gm_2 X}{R_2^3} + \frac{Gm_1(X_E - X)}{R_1^3} - \frac{Gm_1 X_E}{D^3} + a_{T_x} \tag{A.6}$$

$$\ddot{Y} = -\frac{Gm_2 Y}{R_2^3} + \frac{Gm_1(Y_E - Y)}{R_1^3} - \frac{Gm_1 Y_E}{D^3} + a_{T_y} \tag{A.7}$$

The restricted three-body problem equations of motion can be referenced to a rotating, Moon-centered coordinate system by using relative motion analysis and accounting for the Coriolis and centripetal accelerations [8]. The accelerations relative to the rotating frame are as follows:

$$\ddot{\vec{r}}_{rot} = \ddot{\vec{r}}_{abs} - \dot{\vec{\omega}} \times \vec{r}_{rot} - \vec{\omega} \times \vec{\omega} \times \vec{r}_{rot} - 2\vec{\omega} \times \dot{\vec{r}}_{rot} \tag{A.8}$$

After performing the vector products, equation (A.8) becomes:

$$\ddot{X}_{rot} = \ddot{X}_{abs} + 2\dot{Y}_{rot}\omega + Y_{rot}\dot{\omega} + X_{rot}\omega^2 \tag{A.9}$$

$$\ddot{Y}_{rot} = \ddot{Y}_{abs} - 2\dot{X}_{rot}\omega - X_{rot}\dot{\omega} + Y_{rot}\omega^2 \tag{A.10}$$

where

$\ddot{X}_{abs}, \ddot{Y}_{abs}$: absolute acceleration
$\ddot{X}_{rot}, \ddot{Y}_{rot}$: relative acceleration (with respect to the rotating frame)
$\dot{X}_{rot}, \dot{Y}_{rot}$: relative velocity (with respect to the rotating frame)
X_{rot}, Y_{rot} : relative position (with respect to the rotating frame)
ω : angular rate of the rotating frame
$\dot{\omega}$: angular acceleration of the rotating frame

The new Moon-centered frame is defined with the x-axis along the Earth-Moon line and the positive x-direction pointing toward the Earth. The frame rotates with a constant angular rate, ω, as required for the restricted three-body problem and therefore the terms including angular acceleration are not present. The components of the absolute acceleration, $\ddot{X}_{abs}$ and $\ddot{Y}_{abs}$, can be replaced by substituting equations (A.6) and (A.7), the absolute acceleration of the spacecraft in a Moon-referenced frame due to gravitational and thrust forces. The coordinates of the Earth no longer need to be included in equations (A.6) and (A.7) since the x-axis rotates with the

Earth-Moon line; therefore the x-coordinate of the Earth, X_E, is replaced with D and the Earth's y-coordinate, Y_E, is set to zero. The equations of motion with respect to the rotating frame can be rewritten, after dropping the subscripts, as follows:

$$\ddot{X} = -\frac{Gm_2X}{R_2^3} - \frac{Gm_1(X-D)}{R_1^3} - \frac{Gm_1D}{D^3} + a_{T_x} + 2\dot{Y}\omega + X\omega^2 \qquad \text{(A.11)}$$

$$\ddot{Y} = -\frac{Gm_2Y}{R_2^3} - \frac{Gm_1Y}{R_1^3} + a_{T_y} - 2\dot{X}\omega + Y\omega^2 \qquad \text{(A.12)}$$

This fourth-order system of equations can be made dimensionless by defining the reference unit length as the mean Earth-Moon distance, the reference unit time as the inverse of the angular rate of the Earth-Moon system, and the reference unit mass as the total mass of the Earth-Moon system as previously defined in chapter 2 and expressed by equations (2.27) - (2.29).

The resulting dimensionless equations of motion for the restricted three-body problem with thrust terms in a Moon-centered, rotating, Cartesian frame are:

$$\ddot{x} = -\frac{\mu_2 x}{r_2^3} - \frac{\mu_1(x-1)}{r_1^3} - \mu_1 + \tilde{a}_{T_x} + 2\dot{y} + x \qquad \text{(A.13)}$$

$$\ddot{y} = -\frac{\mu_2 y}{r_2^3} - \frac{\mu_1 y}{r_1^3} + \tilde{a}_{T_y} - 2\dot{x} + y \qquad \text{(A.14)}$$

The parameters $\tilde{a}_{T_x}$ and $\tilde{a}_{T_y}$ are the dimensionless thrust acceleration components. The parameters μ_1 and μ_2 correspond to the dimensionless gravitational parameters of the primaries. The above dimensionless equations are referenced in chapter 2 by equations (2.40) and (2.41).

The objective is to formulate the restricted three-body body problem with thrust terms in a rotating, Moon-centered, polar coordinate system. The dimensionless radial and circumferential accelerations in rotating, Moon-centered, polar coordinates are calculated using equations (A.13) and (A.14) and the Moon-centered polar angle, θ_M, as follows:

$$\tilde{a}_r = \ddot{x}\cos\theta_M + \ddot{y}\sin\theta_M \tag{A.15}$$

$$\tilde{a}_{\theta_M} = -\ddot{x}\sin\theta_M + \ddot{y}\cos\theta_M \tag{A.16}$$

Equations (A.15) and (A.16) express the dimensionless radial and circumferential acceleration of the spacecraft due to the restricted three-body problem dynamics, the low-thrust propulsion system, and the transformation from an inertial frame to a rotating frame. The kinematic equations of a particle's acceleration in polar coordinates can be equated to the dimensionless acceleration components, equations (A.15) and (A.16) as shown below:

$$\ddot{r} - r\dot{\theta}_M^2 = \tilde{a}_r \tag{A.17}$$

$$r\ddot{\theta}_M + 2\dot{r}\dot{\theta}_M = \tilde{a}_{\theta_M} \tag{A.18}$$

This fourth-order system can be reduced to four first-order differential equations by introducing the state variables:

$x_1 = r$ (dimensionless Moon-spacecraft position)
$x_2 = \dot{r}$ (dimensionless radial velocity in rotating frame)
$x_3 = r\dot{\theta}_M$ (dimensionless circumferential velocity in rotating frame)
$x_4 = \theta_M$ (Moon polar angle)

The resulting first-order equations of motion are:

$$\dot{x}_1 = x_2 \tag{A.19}$$

$$\dot{x}_2 = \frac{x_3^2}{x_1} + \tilde{a}_r \tag{A.20}$$

$$\dot{x}_3 = -\frac{x_2 x_3}{x_1} + \tilde{a}_{\theta_M} \tag{A.21}$$

$$\dot{x}_4 = \frac{x_3}{x_1} \tag{A.22}$$

The fourth state equation is uncoupled from the system but is required for the transformation from the Cartesian coordinates to the polar coordinates. These four equations are numerically integrated to determine the motion of a thrusting spacecraft in a Moon-centered, rotating frame in the context of the restricted three-body problem.

APPENDIX B. OPTIMIZATION PROGRAM LISTINGS

This appendix lists the driver program for the sequential quadratic programming code and the functional evaluation subroutine that performs the numerical integration of the equations of motion and the explicit computation of the performance index and the constraints. The program corresponding to the optimal Earth-Moon trajectory is listed in this appendix.

```
C
C   PROGRAM FOR RESTRICTED 3-BODY EARTH-MOON TRAJECTORIES
C
C  ***  USES BODY-CENTERED, ROTATING, POLAR COORDINATES   ***
C  ***  FOR THE INTEGRATION OF THE EQUATIONS OF MOTION    ***
C
C ********************************************************
C
C     MINIMIZE  :   TOTAL ENGINE-ON TIME
C
C     SUBJECT TO :  FINAL LUNAR ORBIT CONDITIONS:
C
C                   PARKING  ALT. = 100 km
C                   Vr(TF)      = 0
C                   Vtheta(TF) = CIRCULAR ORBITAL SPEED AT 100 km
C
C     FREE END-TIME PROBLEM  : TF ~ 7 DAYS
C
C                            [ STEERING ANGLES 1-6 (EARTH SPIRAL) ]
```

```
C                             | INITIAL POLAR ANGLE                  |
C     DESIGN VECTOR :   X =  | ENGINE STOP TIME (EARTH SPIRAL)      |
C                             | ENGINE START TIME (MOON SPIRAL)      |
C                             | STEERING ANGLES 1-11 (MOON SPIRAL) |
C                             [ ALPHA, TIME TRANSFORMATION           ]
C
C *******************************************************
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION MU1, MU2, MASFIN
      DOUBLE PRECISION ISP,MDOT,MASSO,MASSF,MASSIC,MASSI,MASS
C
C DEFINE STORAGE REQUIREMENTS.
C
      PARAMETER (MAXF=2  )
      PARAMETER (MAXX=3000)
      PARAMETER (MAXH=20 )
      PARAMETER (MAXG=15 )
      PARAMETER (MAXIO=12)
      PARAMETER (MAXINT = 9001  )
      PARAMETER (MAXCNT = 18001  )
      DIMENSION N(6),IO(MAXIO)
      DIMENSION X(MAXX),F(MAXF),H(MAXH),G(MAXG)
      DIMENSION MD(20)
C
C COMMON BLOCKS
C
      COMMON/STATE/ Y1(MAXINT),Y2(MAXINT),Y3(MAXINT),Y4(MAXINT)
      COMMON/CNTRL/ U(MAXCNT), UPRNT(MAXINT)
      COMMON/SHIP/  THRUST,ISP,MDOT,MASSO,MASS(MAXINT),MASFIN
      COMMON/PERFM/  FMIN, TESC, TCAPT
      COMMON/CONST/ RDIST,WSYN,RM1,RM2,MU1,MU2,REARTH,RMOON
     &         ,R2D,D2R,GMM,GME,GRAV,PI,DREF,TREF,VREF,AREF
     &         ,VMREF,TMREF,AMREF,TSMALL,ALTPRK,ALTLUN
      COMMON/PARAM/  TOF,DT1SEC,DT2SEC,DT3SEC,TF,TAUS(MAXINT)
      COMMON/RELE/ RE(MAXINT),VRE(MAXINT),VTE(MAXINT),
     &             THETAE(MAXINT),ENERGE(MAXINT), ECCENE(MAXINT)
      COMMON/RELM/ RADM(MAXINT), VRADM(MAXINT), VTHETM(MAXINT),
     &             THETM(MAXINT), ENERGM(MAXINT), ECCENM(MAXINT),
     &             HAMOON,HPMOON
```

```
      COMMON/INTPAR/ NINT,NINT1,NINT2,NINT3,IEND,IEND1,IEND2
      COMMON/TIMLIN/  XIN1(21),XOUT1(MAXCNT),XIN2(21),
     &                XOUT2(MAXCNT)
      INTEGER IPNT1(25), IPNT2(25), IPNT3(25), IHDR(25)
      DIMENSION DVAL(25)
      CHARACTER*40 CHVAL(25)
      DATA IPNT1 / 7,16,5,6,19,20,21,9,10,8,12, 14*0 /
      DATA IPNT2 / 7,16,5,6,19,22,23,24,8,12, 15*0 /
      DATA IPNT3 / 7,16,6,22,23,24,9,10,8,12, 15*0 /
      DATA IHDR   / 13,17,14,15, 21*0 /
      DATA NHOUT  /  4  /
      DATA NOUT1  / 11  /
      DATA NOUT2  / 10  /
      DATA NOUT3  / 10  /
      DATA CHVAL(1) /'TOTAL FLIGHT TIME, days             :'/
      DATA CHVAL(2) /'NUMBER OF INTEGRATION STEPS         :'/
      DATA CHVAL(3) /'TOTAL ENGINE-ON TIME, days          :'/
      DATA CHVAL(4) /'FINAL S/C MASS IN LUNAR ORBIT, kg :'/
      DATA CHVAL(5) /'FUEL REQD (% OF INITIAL MASS)       :'/
      DATA CHVAL(6) /'EARTH ESCAPE SPIRAL TIME, days      :'/
      DATA CHVAL(7) /'DISTANCE FROM EARTH, Rearth         :'/
      DATA CHVAL(8) /'EARTH REL. ENERGY, km**2/s**2       :'/
      DATA CHVAL(9) /'EARTH REL. ECCENTRICITY             :'/
      DATA CHVAL(10)/'MOON CAPTURE SPIRAL TIME, days      :'/
      DATA CHVAL(11)/'DISTANCE FROM MOON, Rmoon           :'/
      DATA CHVAL(12)/'THETA (wrt MOON), deg               :'/
      DATA CHVAL(13)/'THETA (wrt EARTH), deg              :'/
      DATA CHVAL(14)/'R-DOT, km/s                         :'/
      DATA CHVAL(15)/'VTHETA, km/s                        :'/
      DATA CHVAL(16)/'MOON REL. ENERGY, km**2/s**2        :'/
      DATA CHVAL(17)/'MOON REL. ECCENTRICITY              :'/
      DATA CHVAL(18)/'START OF CAPTURE SPIRAL, Rmoon      :'/
      DATA CHVAL(19)/'FINAL S/C MASS, kg                  :'/
      DATA CHVAL(20)/'TIME, sec                           :'/
      DATA CHVAL(21)/'TIME, days                          :'/
      DATA CHVAL(22)/'PERILUNE ALTITUDE, km               :'/
      DATA CHVAL(23)/'APOLUNE  ALTITUDE, km               :'/
      DATA CHVAL(24)/'MOON STEERING ANGLE, deg            :'/
      NAMELIST/INPT/ IOPT,TOF,DT1SEC,DT2SEC,DT3SEC,ISTEP1,ISTEP2,
     &               ISTEP3,ALTPRK,ALTLUN,THRUST,MASS0,ISP
```

```
C
      EXTERNAL EVAL
C
C SET INPUT/OUTPUT PARAMETERS.
C
      IPFLAG=1
      IPRINT=2
      ICARD=2
      IIN=5
      IOUT=6
      IPUNCH=7
      ISCALE=2
      IO(1)=IPFLAG
      IO(2)=IPRINT
      IO(6)=ICARD
      IO(7)=IIN
      IO(8)=IOUT
      IO(9)=IPUNCH
      IO(11)=ISCALE
C
C DEFINE VECTOR STORAGE SIZE
C
      N(4)=MAXX
      N(5)=MAXH
      N(6)=MAXG
C
C DEFINE PROBLEM SIZE
C
      NX  = 21
      NH  = 3
      NG  = 0
      N(1) = NX
      N(2) = NH
      N(3) = NG
C
C INPUT NAMELIST IF ANY
C
C
C   READ IN THE INPUT FILE
C
```

```
      READ(14,INPT)
C
C  CONSTANTS ...
C
      PI     = 3.141592654 D0
      R2D    = 180.D0/PI
      D2R    = 1.D0/R2D
      GME    = 3.986011875 D5
      GRAVO  = GME/( (REARTH + ALTPRK)**2 )
      GRAV   = 9.80665 D0
      RDIST  = 384400.D0
      RM1    = 4670.71094 D0
      RM2    = RDIST - RM1
      WSYN   = DSQRT( GME*( 1.D0 + RM1/RM2 )/(RDIST**3) )
      MU1    = RM2/RDIST
      MU2    = RM1/RDIST
      GMM    = GME * (RM1/RM2)
      REARTH = 6378.14453125 D0
      RMOON  = 1738.D0
      RSOI   = 66300.D0
      DREF   = RDIST
      TREF   = 1.D0/WSYN
      VREF   = RDIST*WSYN
      AREF   = 1.D3 * RDIST*(WSYN**2)
      VMREF  = DSQRT( GMM/RMOON )
      TMREF  = RMOON/VMREF
      AMREF  = 1.D3 * GMM/(RMOON**2)
      TSMALL = 1. D-3
      TN2D   = TREF/8.64 D4
      FT2KM  = 3.048D-4
C
C READ IN INPUT DATA FILE
C
      CALL SQP(EVAL,N,X,F,G,H,IO,MD,0)
C
C  CHECK "IOPT" SWITCH : IF IOPT > 0 , PERFORM OPTIMIZATION
C
      IF( IOPT .EQ. 0 ) GO TO 99
      INOM = 0
      GO TO 99
```

```
 747  CONTINUE
      INOM = 1
C
C PERFORM THE OPTIMIZATION.
C
      CALL SQP(EVAL,N,X,F,G,H,IO,MD,-1)
  99  CONTINUE
C
C  UNSCALE THE DESIGN VECTOR, X
C
      DO 13 I=1,NX
      X(I) = X(I)*X(I+NX)
  13  CONTINUE
C
C  CALL EVAL TO FIND THE OPTIMAL TRAJECTORY USING THE OPTIMAL
C  CONTROL AS DETERMINED BY SQP
C
      CALL EVAL(N,X,F,G,H,IO,IER)
C
C OUTPUT THE OPTIMAL CONTROL PARAMETER VECTOR, THE OPTIMAL
C CONTROL HISTORIES, AND THE CORRESPONDING OPTIMAL STATE
C TRAJECTORIES.
C
C     DO 2  I=1, NX
C       XDEG = X(I)*R2D
C       IF(I .EQ. 4 .OR. I .GT. 5) XDEG = X(I)
C   2   WRITE(10,*) XDEG
C     DO 3  I=1, IENDD, 2
C       CONTRD = U(I) * R2D
C   3   WRITE(11,*) CONTRD
C
C OUTPUT THE STATE TRAJECTORY AND THE ORBITAL ELEMENTS...
C
      TFD   = TAUS(IEND)/8.64D4
      TBURN = TESC + TCAPT
      TBURND = TBURN*TN2D
      FUELR = 100.DO*( 1.DO - MASFIN/MASSO )
      IF( IOPT .EQ. O .OR. INOM .EQ. O ) THEN
       WRITE(IOUT,*)
       WRITE(IOUT,*) ' ***  NOMINAL (NON-OPTIMAL) TRAJECTORY  ***'
```

```
         WRITE(IOUT,*)
         GO TO 501
        ENDIF
        WRITE(IOUT,*)
        WRITE(IOUT,*) 'MINIMIZE   :   TOTAL ENGINE-ON TIME'
        WRITE(IOUT,*)
        WRITE(IOUT,*) 'SUBJECT TO :   FINAL LUNAR ORBIT REQMNTS.'
        WRITE(IOUT,*) '               ECCENTRICITY <= 0.01'
        WRITE(IOUT,*) '               PARKING ALT.  = 100 km'
        WRITE(IOUT,*)
 501    WRITE(IOUT,*) ' ********************** RESULT ***********
     &*************'
        WRITE(IOUT,7001) ( CHVAL(1), TFD   )
        WRITE(IOUT,7002) ( CHVAL(2), NINT  )
        WRITE(IOUT,7001) ( CHVAL(3), TBURND )
        WRITE(IOUT,7001) ( CHVAL(4), MASFIN )
        WRITE(IOUT,7001) ( CHVAL(5), FUELR )
        WRITE(IOUT,*) ' *******************************************
     &*************'
        WRITE(IOUT,*)
        WRITE(IOUT,*) ' *********  OPTIMAL DESIGN VECTOR  ***********'
        WRITE(6,*) ( X(I)*R2D , I=1,6 )
        WRITE(6,*) X(7)*R2D, X(8), X(9)
        WRITE(6,*) ( X(I)*R2D , I=10,20 )
        WRITE(6,*) X(21)
        WRITE(IOUT,*) ' ********************************************'
C
C OUTPUT THE BOUNDARY CONDITIONS FROM THE 3-BODY SPIRAL
C
        TESCD  = TESC*TN2D
        TCAPTD = TCAPT*TN2D
C
C  TAB OUTPUT FOR STATE VARIABLES *** EARTH SPIRAL PHASE
C
        WRITE(IOUT,*)
        WRITE(IOUT,*) ' ********************************************
     &****************'
        WRITE(IOUT,*) ' ***************  EARTH ESCAPE SPIRAL PHASE  *
     &****************'
        WRITE(IOUT,*) ' ********************************************
```

```
     &****************'
      DO 401 I=1, IEND1, ISTEP1
      R1 = RE(I)
      CTHET = DCOS( Y4(I) )
      STHET = DSIN( Y4(I) )
      XE = R1*REARTH*CTHET
      YE = R1*REARTH*STHET
      R2 = DSQRT( (XE + DREF)**2 + YE**2 )/RMOON
C
C  ASSIGN THE PARAMETERS TO THE "DVAL" ARRAY
C
      DVAL(1) = Y1(I)                   ! Earth radial dist., ND
      DVAL(2) = Y2(I)                   ! r-dot, rotat. crds, ND
      DVAL(3) = Y3(I)                   ! vthet, rotat. crds, ND
      DVAL(4) = Y4(I)                   ! theta, rotat. crds, ND
      DVAL(5) = R1                      ! earth-S/C dist., Rearth
      DVAL(6) = R2                      ! moon-S/C dist., Rmoon
      DVAL(7) = TAUS(I)                 ! time, sec
      DVAL(8) = UPRNT(I)                ! steering angle, deg
      DVAL(9) = ENERGE(I)               ! Earth rel. energy, km**2/s**2
      DVAL(10) = ECCENE(I)              ! Earth rel. eccentricity
      DVAL(12) = MASS(I)                ! S/C mass , kg
      DVAL(13) = THRUST                 ! thrust, N
      DVAL(14) = ISP                    ! specific impulse, sec
      DVAL(15) = MDOT                   ! mass flow rate, kg/s
      DVAL(16) = TAUS(I)/8.64D4         ! time, days
      DVAL(17) = THRUST/( MASS0*GRAV )  ! initial T/W ratio
      DVAL(19) = THETAE(I)              ! rel. Earth angular pos., deg
      DVAL(20) = VRE(I)                 ! rad. vel., Earth rel, km/s
      DVAL(21) = VTE(I)                 ! circumfr. vel., Earth rel, km/s
      CALL PROUT(ISTRT,IOUT,NHOUT,NOUT1,IHDR,IPNT1,DVAL)
  401 CONTINUE
C
C  ASSIGN THE PARAMETERS TO THE "DVAL" ARRAY (LAST INTEG. STEP)
C
      I = IEND1
      R1 = RE(I)
      CTHET = DCOS( Y4(I) )
      STHET = DSIN( Y4(I) )
      XE = R1*REARTH*CTHET
```

```
      YE = R1*REARTH*STHET
      R2 = DSQRT( (XE + DREF)**2 + YE**2 )/RMOON
      DVAL(1) = Y1(I)                    ! Earth radial dist., ND
      DVAL(2) = Y2(I)                    ! r-dot, rotat. crds, ND
      DVAL(3) = Y3(I)                    ! vthet, rotat. crds, ND
      DVAL(4) = Y4(I)                    ! theta, rotat. crds, ND
      DVAL(5) = R1                       ! earth-S/C dist., Rearth
      DVAL(6) = R2                       ! moon-S/C dist., Rmoon
      DVAL(7) = TAUS(I)                  ! time, sec
      DVAL(8) = UPRNT(I)                 ! steering angle, deg
      DVAL(9) = ENERGE(I)                ! Earth rel. energy, km**2/s**2
      DVAL(10) = ECCENE(I)               ! Earth rel. eccentricity
      DVAL(12) = MASS(I)                 ! S/C mass , kg
      DVAL(13) = THRUST                  ! thrust, N
      DVAL(14) = ISP                     ! specific impulse, sec
      DVAL(15) = MDOT                    ! mass flow rate, kg/s
      DVAL(16) = TAUS(I)/8.64D4          ! time, days
      DVAL(17) = THRUST/( MASSO*GRAV) ! initial T/W ratio
      DVAL(19) = THETAE(I)               ! rel. Earth angular pos., deg
      DVAL(20) = VRE(I)                  ! rad. vel., Earth rel, km/s
      DVAL(21) = VTE(I)                  ! circumfr. vel., Earth rel, km/s
      CALL PROUT(ISTRT,IOUT,NHOUT,NOUT1,IHDR,IPNT1,DVAL)
C
C  TAB OUTPUT FOR STATE VARIABLES *** TRANSLUNAR COAST PHASE
C
      WRITE(IOUT,*) ' ********************************************
     &****************'
      WRITE(IOUT,*) ' ***************   TRANSLUNAR COAST PHASE  ***
     &****************'
      WRITE(IOUT,*) ' ********************************************
     &****************'
      DO 402 I=IEND1, IEND2, ISTEP2
      R1 = RE(I)
      R2 = RADM(I)
C
C  ASSIGN THE PARAMETERS TO THE "DVAL" ARRAY
C
      DVAL(1) = Y1(I)              ! Earth radial dist., ND
      DVAL(2) = Y2(I)              ! r-dot, rotat. crds, ND
      DVAL(3) = Y3(I)              ! vtheta, rotat. crds, ND
```

```
      DVAL(4) = Y4(I)             ! theta, rotat. crds, ND
      DVAL(5) = R1                ! earth-S/C dist., Rearth
      DVAL(6) = R2                ! moon-S/C dist., Rmoon
      DVAL(7) = TAUS(I)           ! time, sec
      DVAL(8) = UPRNT(I)          ! steering angle, deg
      DVAL(12) = MASS(I)          ! S/C mass , kg
      DVAL(16) = TAUS(I)/8.64D4   ! time, days
      DVAL(19) = THETAE(I)        ! rel. Earth angular pos., deg
      DVAL(22) = THETM(I)         ! rel. Moon angular pos., deg
      DVAL(23) = VRADM(I)         ! rad. vel., Moon rel, km/s
      DVAL(24) = VTHETM(I)        ! circumfr. vel., Moon rel, km/s
      CALL PROUT(ISTRT,IOUT,NHOUT,NOUT2,IHDR,IPNT2,DVAL)
  402 CONTINUE
C
C  ASSIGN THE PARAMETERS TO THE "DVAL" ARRAY (LAST INTEG. STEP)
C
      I = IEND2
      R1 = RE(I)
      R2 = RADM(I)
C
C  ASSIGN THE PARAMETERS TO THE "DVAL" ARRAY
C
      DVAL(1) = Y1(I)            ! Earth radial dist., ND
      DVAL(2) = Y2(I)            ! r-dot, rotat. crds, ND
      DVAL(3) = Y3(I)            ! vtheta, rotat. crds, ND
      DVAL(4) = Y4(I)            ! theta, rotat. crds, ND
      DVAL(5) = R1               ! earth-S/C dist., Rearth
      DVAL(6) = R2               ! moon-S/C dist., Rmoon
      DVAL(7) = TAUS(I)          ! time, sec
      DVAL(8) = UPRNT(I)         ! steering angle, deg
      DVAL(12) = MASS(I)         ! S/C mass , kg
      DVAL(16) = TAUS(I)/8.64D4 ! time, days
      DVAL(19) = THETAE(I)       ! rel. Earth angular pos., deg
      DVAL(22) = THETM(I)        ! rel. Moon angular pos., deg
      DVAL(23) = VRADM(I)        ! rad. vel., Moon rel, km/s
      DVAL(24) = VTHETM(I)       ! circumfr. vel., Moon rel, km/s
      CALL PROUT(ISTRT,IOUT,NHOUT,NOUT2,IHDR,IPNT2,DVAL)
C
C  TAB OUTPUT FOR STATE VARIABLES *** MOON CAPTURE SPIRAL PHASE
C
```

```
      WRITE(IOUT,*) ' *********************************************
     &****************'
      WRITE(IOUT,*) ' ***************  MOON CAPTURE SPIRAL PHASE  *
     &****************'
      WRITE(IOUT,*) ' *********************************************
     &****************'
      DO 403 I=IEND2, IEND, ISTEP3
C
C  ASSIGN THE PARAMETERS TO THE "DVAL" ARRAY
C
      DVAL(6) = RADM(I)           ! moon-S/C dist., Rmoon
      DVAL(7) = TAUS(I)           ! time, sec
      DVAL(8) = UPRNT(I)          ! steering angle, deg
      DVAL(9) = ENERGM(I)         ! Moon rel. energy, km**2/s**2
      DVAL(10) = ECCENM(I)        ! Moon rel. eccentricity
      DVAL(12) = MASS(I)          ! S/C mass , kg
      DVAL(16) = TAUS(I)/8.64D4   ! time, days
      DVAL(22) = THETM(I)         ! rel. Moon angular pos., deg
      DVAL(23) = VRADM(I)         ! rad. vel., Moon rel, km/s
      DVAL(24) = VTHETM(I)        ! circumfr. vel., Moon rel, km/s
      CALL PROUT(ISTRT,IOUT,NHOUT,NOUT3,IHDR,IPNT3,DVAL)
  403 CONTINUE
C
      WRITE(IOUT,*)
      WRITE(IOUT,*) ' *****************  BOUNDARY CONDITIONS (t=TF)
     &*************'
      WRITE(IOUT,*) ' *****  END OF 3-BODY SIMULATION  *****'
      WRITE(IOUT,7001) ( CHVAL(20), TAUS(IEND) )
      WRITE(IOUT,7001) ( CHVAL(21), TAUS(IEND)/8.64D4 )
      WRITE(IOUT,7001) ( CHVAL(11), RADM(IEND) )
      WRITE(IOUT,7001) ( CHVAL(12), THETM(IEND) )
      WRITE(IOUT,7001) ( CHVAL(14), VRADM(IEND) )
      WRITE(IOUT,7001) ( CHVAL(15), VTHETM(IEND) )
      WRITE(IOUT,7001) ( CHVAL(16), ENERGM(IEND) )
      WRITE(IOUT,7001) ( CHVAL(17), ECCENM(IEND) )
      WRITE(IOUT,7001) ( CHVAL(22), HPMOON )
      WRITE(IOUT,7001) ( CHVAL(23), HAMOON )
      WRITE(IOUT,7001) ( CHVAL(24), UPRNT(IEND) )
      WRITE(IOUT,7001) ( CHVAL(19), MASS(IEND) )
      WRITE(IOUT,*) ' ********************************************
```

```
     &*************'
C
C
C  OUTPUT FOR PLOTTING  ( TOTAL.PLOT.DAT )
C
C
C  EARTH ESCAPE PLOTTING
C
      DO 5  I=1, IEND1
      XE = RE(I)*DCOS( THETAE(I)*D2R )
      YE = RE(I)*DSIN( THETAE(I)*D2R )
      TAUD = TAUS(I)/8.64D4
      CPHASE = DCOS( WSYN*TAUS(I) )
      SPHASE = DSIN( WSYN*TAUS(I) )
      XI = XE*CPHASE - YE*SPHASE
      YI = XE*SPHASE + YE*CPHASE
      XLUN = (-DREF/REARTH)*CPHASE
      YLUN = (-DREF/REARTH)*SPHASE
      WRITE(15,2112) XI, YI, XLUN, YLUN
  5   WRITE(13,2114) XE, YE
C
C  TRANSLUNAR COAST PLOTTING
C
      DO 51  I=IEND1, IEND2
      XE = RE(I)*DCOS( THETAE(I)*D2R )
      YE = RE(I)*DSIN( THETAE(I)*D2R )
      TAUD = TAUS(I)/8.64D4
      CPHASE = DCOS( WSYN*TAUS(I) )
      SPHASE = DSIN( WSYN*TAUS(I) )
      XI = XE*CPHASE - YE*SPHASE
      YI = XE*SPHASE + YE*CPHASE
      XLUN = (-DREF/REARTH)*CPHASE
      YLUN = (-DREF/REARTH)*SPHASE
      WRITE(15,2112) XI, YI, XLUN, YLUN
  51  WRITE(13,2114) XE, YE
C
C  MOON CAPTURE PLOTTING
C
      DO 52  I=IEND2, IEND
      TAUD = TAUS(I)/8.64D4
```

```
       R2 = RADM(I)*RMOON/REARTH
       XE = R2*DCOS( THETM(I)*D2R ) - DREF/REARTH
       YE = R2*DSIN( THETM(I)*D2R )
       XM = RADM(I)*DCOS( THETM(I)*D2R )
       YM = RADM(I)*DSIN( THETM(I)*D2R )
      CPHASE = DCOS( WSYN*TAUS(I) )
      SPHASE = DSIN( WSYN*TAUS(I) )
      XI = XE*CPHASE - YE*SPHASE
      YI = XE*SPHASE + YE*CPHASE
      XLUN = (-DREF/REARTH)*CPHASE
      YLUN = (-DREF/REARTH)*SPHASE
      WRITE(15,2112) XI, YI, XLUN, YLUN
 52   WRITE(13,2114) XE, YE
      WRITE(6,*) ' IEND1 = ',IEND1,' IEND2 = ',IEND2,' IEND = ',IEND
c      DO 6  I=1, IEND2
c      TAUD = TAUS(I)/8.64D4
c  6   WRITE(13,2113) TAUD, vre(i), vte(i)
c      DO 7  I=iend2, IEND
c      TAUD = TAUS(I)/8.64D4
c  7   WRITE(13,2113) TAUD, vradm(i), vthetm(i)
 2112 FORMAT(1X,4E15.6)
 2113 FORMAT(1X,3E15.6)
 2114 FORMAT(1X,2E15.6)
 7001 FORMAT(2X,A40,1X,F15.6)
 7002 FORMAT(2X,A40,1X,I15)
C
      IF( INOM .EQ. 0 .AND. IOPT .GT. 0 ) GO TO 747
      STOP
      END
C
C
      SUBROUTINE EVAL(N,X,F,G,H,IO,IER)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION MU1, MU2, MASFIN
      DOUBLE PRECISION ISP,MDOT,MASSO,MASSF,MASSIC,MASSI,MASS
      DIMENSION N(*),X(*),F(*),G(*),H(*),IO(*)
      DIMENSION XU(6), XUM(11)
      PARAMETER (MAXINT = 9001  )
      PARAMETER (MAXCNT = 18001  )
```

```
      DIMENSION  D(10),DS(10),FJ(101),VS(10),XS(10),Y(10)
C
C COMMON BLOCKS
C
      COMMON/STATE/ Y1(MAXINT),Y2(MAXINT),Y3(MAXINT),Y4(MAXINT)
      COMMON/CNTRL/ U(MAXCNT), UPRNT(MAXINT)
      COMMON/SHIP/  THRUST,ISP,MDOT,MASSO,MASS(MAXINT),MASFIN
      COMMON/PERFM/  FMIN, TESC, TCAPT
      COMMON/CONST/ RDIST,WSYN,RM1,RM2,MU1,MU2,REARTH,RMOON
     &          ,R2D,D2R,GMM,GME,GRAV,PI,DREF,TREF,VREF,AREF
     &          ,VMREF,TMREF,AMREF,TSMALL,ALTPRK,ALTLUN
      COMMON/PARAM/  TOF,DT1SEC,DT2SEC,DT3SEC,TF,TAUS(MAXINT)
      COMMON/RELE/ RE(MAXINT),VRE(MAXINT),VTE(MAXINT),
     &             THETAE(MAXINT),ENERGE(MAXINT), ECCENE(MAXINT)
      COMMON/RELM/ RADM(MAXINT), VRADM(MAXINT), VTHETM(MAXINT),
     &             THETM(MAXINT), ENERGM(MAXINT), ECCENM(MAXINT),
     &             HAMOON,HPMOON
      COMMON/INTPAR/ NINT,NINT1,NINT2,NINT3,IEND,IEND1,IEND2
      COMMON/TIMLIN/  XIN1(21),XOUT1(MAXCNT),XIN2(21),
     &                XOUT2(MAXCNT)
C
      IPRINT = IO(2)
      IOUT = IO(8)
      NX = N(1)
      NH = N(2)
      NG = N(3)
      IER = 0
      IGO = 1
      IF (IO(12) .EQ. 0)    IGO = 7
C
C OUTPUT THE CONTROL PARAMETER VECTOR
C
      IF (IPRINT .GE. 5) THEN
          WRITE(IOUT,9000) (I,X(I),I=1,NX)
      ELSE
          IF (IGO .EQ. 7) THEN
              IF (IPRINT .GE. 3 .AND. IPRINT .LE. 4) THEN
                   WRITE(IOUT,9000) (I,X(I),I=1,NX)
              END IF
           END IF
```

```
      END IF
C
C SET PARAMETER VALUES
C
C
C  SET ENGINE-ON/OFF TIMES AND TIME TRANSFORMATION VARIABLE
C
      TOFF  = X(8)
      TON   = X(9)
      ALPHA = X(21)
      TF    = TON + ALPHA*( TOF/TREF - TON )
      TCAPT1 = ALPHA*( TOF/TREF - TON )
C
C  CALCULATE THE # OF INTEGRATION STEPS FOR
C  EACH INTEGRATION ARC BASED ON EACH DT
C
      DT1   = DT1SEC/TREF
      DT2   = DT2SEC/TREF
      DT3   = DT3SEC/TREF
      NINT1 = DINT( TOFF/DT1 )
      NINT2 = DINT( (TON - TOFF)/DT2 )
      NINT3 = DINT( (TF  - TON )/DT3 )
      NINT  = NINT1 + NINT2 + NINT3
C
C  ADJUST THE TIME STEPS FOR THE CALCULATED NINT'S
C
      DT1 = TOFF/DFLOAT(NINT1)
      DT2 = (TON - TOFF)/DFLOAT(NINT2)
      DT3 = (TF  - TON )/DFLOAT(NINT3)
C
C  SET FLAGS FOR THE SPLINE FITS
C
      NINT1D = 2*NINT1
      NINT3D = 2*NINT3
      IEND1  = NINT1 + 1
      IEND2  = NINT1 + NINT2 + 1
      IEND   = NINT + 1
      IEND1D = NINT1D + 1
      IEND3D = NINT3D + 1
C
```

```
C ****************************************************
C *************  FIRST INTEGRATION SEGMENT   **************
C *************    EARTH ESCAPE SPIRAL       **************
C ****************************************************
C
C ****************************************************
C ***********  INITIAL CONDITIONS OF THE S/C  *************
C ****************************************************
C
C  STATE VECTOR I.C. ( WRT GEOCENTRIC, ROTATING POLAR COORD.'S )
C  ( NON-DIMENSIONAL )
C
C   INPUT IN FIXED, GEOCENTRIC POLAR COORDINATES AND TRANSFER TO
C   ROTATING GEOCENTRIC POLAR COORD'S
C
C   INPUTS : R , THETA ==> POSITION OF S/C ( THETA WRT XROT )
C            VR, VTHET ==> RADIAL AND CIRCUMFERENTIAL VELOCITY
C
      R      = ALTPRK + REARTH
      VR     = 0.D0
      VTHETA = DSQRT( GME/R )
      THETA  = X(7)
      THETAD = THETA * R2D
C
C  SAVE INITIAL TIME POINT FOR PRINTOUT
C
      RE(1)     = R/REARTH
      VRE(1)    = VR
      VTE(1)    = VTHETA
      THETAE(1) = THETAD
      VRELE     = DSQRT( VR**2 + VTHET**2 )
      ENERGE(1) = VRELE**2/2.D0 - GME/R
      ECCENE(1) = 0.D0
C
C  TRANSFER TO ROTATING GEOCENTRIC COORDS. AND NON-DIMENSIONALIZE
C
C   Y1 = EARTH TO S/C RADIUS
C   Y2 = RADIAL VELOCITY
C   Y3 = CIRCUMFR. VELOCITY (ROTATING COORDS)
C   Y4 = POLAR ANGLE
```

```
C
      Y1(1)   = R/DREF
      Y2(1)   = VR/VREF
      Y3(1)   = ( VTHETA - WSYN*R )/VREF
      Y4(1)   = THETA
C
C  I.C. AT L4
C
c      Y1(1)   = 1.D0
c      Y2(1)   = 0.D0
c      Y3(1)   = 0.D0
c      Y4(1)   = 2.D0*PI/3.D0
C
C   ***  STATE VECTOR I.C.  ***
C
      Y(1)   = Y1(1)
      Y(3)   = Y3(1)
      Y(2)   = Y2(1)
      Y(4)   = Y4(1)
C      WRITE(6,*)
C      WRITE(6,*) 'INSIDE EVAL : '
C      WRITE(6,*) ' Y(1&2) :',Y(1),Y(2)
C      WRITE(6,*) ' Y(3&4) :',Y(3),Y(4)
C
C  SET TIMELINES FOR LATER CONTROL HISTORY INTERPOLATIONS
C  ( EARTH ESCAPE SPIRAL )
C
      NXU  = 6
      DO 4 I=1,NXU
    4  XU(I) = X(I)
      XIN1(1) = 0.D0
      XIN1(NXU) = TOFF
      NXUM1 = NXU - 1
      DELT = TOFF/DFLOAT(NXUM1)
      DO 5  I=2, NXUM1
    5   XIN1(I) = XIN1(I-1) + DELT
      XOUT1(1) = 0.D0
      XOUT1(IEND1D) = TOFF
      HDT = 0.5D0*DT1
      DO 6  I=2, NINT1D
```

```
    6   XOUT1(I) = XOUT1(I-1) + HDT
C
C INTERPOLATE AMONG THE CONTROL POINTS TO OBTAIN A CONTROL TIME
C HISTORY SUITABLE FOR THE NUMERICAL INTEGRATION WHICH FOLLOWS.
C THERE ARE TWICE AS MANY CONTROL ENTRIES AS STATE ENTRIES TO
C ACCOMMODATE THE MIDPOINT EVALUATIONS REQUIRED BY THE RUNGE-KUTTA
C INTEGRATION.
C
      CALL SPLINE(NXU,XIN1,XU,IEND1D,XOUT1,U,IERR)
C
C  RE-DEFINE THE INITAL MASS (MASSIC, kg)
C
      MASSIC  = MASSO
      MASS(1) = MASSIC
C
C  ENGINE FLOW RATE
C
      MDOT = THRUST/(ISP*GRAV)
C
C *************************************************************
C INTEGRATE THE STATE EQUATIONS FOR THE CURRENT CONTROL HISTORY.
C *************************************************************
C
      NV  = 4
      DT  = DT1
      TAU = 0.DO
      J   = 1
      JC  = 1
      IN  = 1
      CONTRL = U(1)
      UPRNT(1) = CONTRL*R2D
      GO TO 103
  101 JP1  = J + 1
      JCP1 = JC + 1
      TAU  = TAU + DT
      K = JC + JC
      CONTRL = U(K)
      GO TO 103
  102 K = K + 1
      CONTRL = U(K)
```

```
  103 CONTINUE
  104 YONE = Y(1)
      YTWO = Y(2)
      YTHR = Y(3)
      YFOR = Y(4)
C
C  CALCULATE THE SINE AND COSINE OF THE ANGLE FROM THE ROTATING
C  X-AXIS TO THE R1 VECTOR
C
      STHETA = DSIN(YFOR)
      CTHETA = DCOS(YFOR)
C
C  CALCULATE THE DISTANCES FROM THE EARTH TO THE S/C AND
C  THE MOON TO THE S/C
C
      R1 = YONE
      XE = R1*CTHETA
      YE = R1*STHETA
      R2 = DSQRT( (XE + 1.D0)**2 + YE**2 )
C
C  UPDATE THE CURRENT S/C MASS.
C  CALCULATE THE THRUST ACCELERATION IN km/s**2 AND CONVERT
C  TO DIMENSIONLESS UNITS OF DU/TU**2
C
      MASSI  = MASSIC - MDOT*TAU*TREF
      ATHRUS = (THRUST/MASSI)/AREF
C
C  RADIAL & CIRCUMFERENTIAL THRUST ACCELERATIONS
C
      ATRAD   = ATHRUS * DSIN( CONTRL )
      ATTHET  = ATHRUS * DCOS( CONTRL )
C
C  CONVERT RADIAL AND CIRCUMFERENTIAL THRUST ACCL'S TO
C  INERTIAL COORD'S
C
      ATX = ATRAD*CTHETA - ATTHET*STHETA
      ATY = ATRAD*STHETA + ATTHET*CTHETA
C
C  CALCULATE THE ACCEL'N DUE TO THE POTENTIAL FUNCTION (GRAVITY
C  FIELD) IN THE INERTIAL COORD'S
```

```
C
      OMEGAX = -( ( MU1*XE/(R1**3) ) +
     &            ( MU2*(XE+1.D0)/(R2**3) ) ) + MU2
      OMEGAY = -((MU1*YE/(R1**3)) + (MU2*YE/(R2**3)))
C
C  TOTAL RADIAL AND CIRCUMFR.  ACCEL. IN FIXED, GEOCENTRIC X-Y COORD.
C
      XDE    = YTWO*CTHETA - YTHR*STHETA
      YDE    = YTWO*STHETA + YTHR*CTHETA
      XDD    = OMEGAX + ATX + 2.D0*YDE + XE
      YDD    = OMEGAY + ATY - 2.D0*XDE + YE
C
C  ACCEL. IN FIXED, GEOCENTRIC POLAR COORDS.
C
      ARAD  =  XDD*CTHETA + YDD*STHETA
      ATHET = -XDD*STHETA + YDD*CTHETA
C
C  DIFF. EQ'S OF MOTION ...
C
      D(1) = YTWO
      D(2) = ARAD + YTHR**2/YONE
      D(3) = ATHET - YTWO*YTHR/YONE
      D(4) = YTHR/YONE
      GO TO (1001,1005,1007,1009,1011), IN
  105 J  = JP1
      JC = JCP1
      Y1(J) = YONE
      Y2(J) = YTWO
      Y3(J) = YTHR
      Y4(J) = YFOR
      TAUS(J) = TAU*TREF
      MASS(J) = MASSI
      UPRNT(J) = CONTRL * R2D
C
C  ***  FIXED, GEOCENTRIC POLAR COORD. CALCULATIONS  ***
C
      RE(J) = YONE*DREF/REARTH
C
C  CALCULATE FIXED EARTH REL. RADIAL & CIRCUMF. VELOCITY (km/s)
C
```

```
      VRE(J)    = YTWO*VREF
      VTE(J)    = ( YTHR + YONE )*VREF
      THETAE(J) = DACOS( CTHETA )*R2D
      IF( STHETA .LT. 0.D0 ) THETAE(J) = 360.D0 - THETAE(J)
      VRELE     = DSQRT( VRE(J)**2 + VTE(J)**2 )
      ENERGE(J) = VRELE**2/2.D0 - GME/(YONE*DREF)
      ANGME     = YONE*DREF*VTE(J)
      ECCENE(J) = DSQRT(1.D0+(2.D0*ENERGE(J)*ANGME**2)/(GME**2))
      IF ( J .LT. IEND1 )    GO TO 1003
C
C  GO TO SECOND INTEGRATION ARC ( TRANSLUNAR COAST )
C
      JPAST = J
      GO TO 222
C
C STANDARD FOURTH-ORDER RUNGE-KUTTA INTEGRATION CODE FOLLOWS
C  ( FOR FIRST INTEGRATION ARC )
C
 1001 H2 = 0.5D0*DT
      H6 = DT/6.D0
      DO 1002  I=1, NV
        DS(I) = D(I)
 1002   XS(I) = Y(I)
 1003 DO 1004  I=1, NV
 1004   Y(I) = XS(I) + H2*DS(I)
      IN = 2
      GO TO 101
 1005 DO 1006  I=1, NV
        DD = D(I)
        VS(I) = DS(I) + 2.D0*DD
 1006   Y(I) = XS(I) + H2*DD
      IN = 3
      GO TO 104
 1007 DO 1008  I=1, NV
        DD = D(I)
        VS(I) = VS(I) + 2.D0*DD
 1008   Y(I) = XS(I) + DT*DD
      IN = 4
      GO TO 102
 1009 DO 1010  I=1, NV
```

```
       VSS = H6*( VS(I) + D(I) )
       XS(I) = XS(I) + VSS
 1010  Y(I) = XS(I)
      IN = 5
      GO TO 104
 1011 DO 1012  I=1, NV
 1012  DS(I) = D(I)
      GO TO 105
C
  222 CONTINUE
C
C  *********************************************************
C  *************  SECOND INTEGRATION SEGMENT  **************
C  ************* TRANSLUNAR COAST, ENGINE OFF **************
C  *********************************************************
C
C   ***  STATE VECTOR I.C.  ( USE PRIOR STATE ) ***
C
      Y(1)  = Y1(JPAST)
      Y(2)  = Y2(JPAST)
      Y(3)  = Y3(JPAST)
      Y(4)  = Y4(JPAST)
C
C  RE-DEFINE THE INITAL MASS (MASSIC, kg) USING THE PRIOR
C  MASS VALUE FROM THE FIRST ARC
C
      MASSIC = MASSI
C
C ************************************************************
C INTEGRATE THE STATE EQUATIONS FOR THE CURRENT CONTROL HISTORY.
C ************************************************************
C
      NV = 4
      DT = DT2
      J  = JPAST
      IN = 1
      CONTRL = 0.DO
      GO TO 203
  201 JP1  = J + 1
      TAU  = TAU + DT
```

```
      GO TO 203
  202 CONTINUE
  203 CONTINUE
  204 YONE = Y(1)
      YTWO = Y(2)
      YTHR = Y(3)
      YFOR = Y(4)
C
C  CALCULATE THE SINE AND COSINE OF THE ANGLE FROM THE ROTATING
C  X-AXIS TO THE R1 VECTOR
C
      STHETA = DSIN(YFOR)
      CTHETA = DCOS(YFOR)
C
C  CALCULATE THE DISTANCES FROM THE EARTH TO THE S/C AND
C  THE MOON TO THE S/C
C
      R1 = YONE
      XE = R1*CTHETA
      YE = R1*STHETA
      R2 = DSQRT( (XE + 1.DO)**2 + YE**2 )
C
C  FIX THE CURRENT S/C MASS.
C
      MASSI  = MASSIC
      ATHRUS = 0.DO
C
C  RADIAL & CIRCUMFERENTIAL THRUST ACCELERATIONS
C
      ATRAD   = ATHRUS * DSIN( CONTRL )
      ATTHET  = ATHRUS * DCOS( CONTRL )
C
C  CONVERT RADIAL AND CIRCUMFERENTIAL THRUST ACCL'S TO
C  INERTIAL COORD'S
C
      ATX = ATRAD*CTHETA - ATTHET*STHETA
      ATY = ATRAD*STHETA + ATTHET*CTHETA
C
C  CALCULATE THE ACCEL'N DUE TO THE POTENTIAL FUNCTION (GRAVITY
C  FIELD) IN THE INERTIAL COORD'S
```

```
C
      OMEGAX = -( ( MU1*XE/(R1**3) ) +
     &            ( MU2*(XE+1.D0)/(R2**3) ) ) + MU2
      OMEGAY = -((MU1*YE/(R1**3)) + (MU2*YE/(R2**3)))
C
C  TOTAL RADIAL AND CIRCUMFR.  ACCEL. IN FIXED, GEOCENTRIC X-Y COORD.
C
      XDE   = YTWO*CTHETA - YTHR*STHETA
      YDE   = YTWO*STHETA + YTHR*CTHETA
      XDD   = OMEGAX + ATX + 2.D0*YDE + XE
      YDD   = OMEGAY + ATY - 2.D0*XDE + YE
C
C  ACCEL. IN FIXED, GEOCENTRIC POLAR COORDS.
C
      ARAD  =  XDD*CTHETA + YDD*STHETA
      ATHET = -XDD*STHETA + YDD*CTHETA
C
C  DIFF. EQ'S OF MOTION ...
C
      D(1) = YTWO
      D(2) = ARAD + YTHR**2/YONE
      D(3) = ATHET - YTWO*YTHR/YONE
      D(4) = YTHR/YONE
      GO TO (2001,2005,2007,2009,2011), IN
  205 J  = JP1
      JC = JCP1
      Y1(J) = YONE
      Y2(J) = YTWO
      Y3(J) = YTHR
      Y4(J) = YFOR
      TAUS(J) = TAU*TREF
      MASS(J) = MASSI
      UPRNT(J) = 0.D0
C
C  ***  FIXED, GEOCENTRIC POLAR COORD. CALCULATIONS  ***
C
      RE(J) = YONE*DREF/REARTH
C
C  CALCULATE FIXED EARTH REL. RADIAL & CIRCUMF. VELOCITY (km/s)
C
```

```
      VRE(J)     = YTWO*VREF
      VTE(J)     = ( YTHR + YONE )*VREF
      THETAE(J) = DACOS( CTHETA )*R2D
      IF( STHETA .LT. 0.D0 ) THETAE(J) = 360.D0 - THETAE(J)
      VRELE      = DSQRT( VRE(J)**2 + VTE(J)**2 )
      ENERGE(J) = VRELE**2/2.D0 - GME/(YONE*DREF)
      ANGME      = YONE*DREF*VTE(J)
      ECCENE(J) = DSQRT(1.D0+(2.D0*ENERGE(J)*ANGME**2)/(GME**2))
C
C  CONVERT THE STATE VECTOR FROM THE FIXED GEOCENTRIC POLAR COORD.
C  SYSTEM TO A FIXED MOON-CENTERED POLAR COORD. SYSTEM FOR PRINTOUT
C
C
C  CALCULATE MOON RELATIVE DISTANCES ( km )
C
      XMOON   = ( YONE*CTHETA + 1.D0 )*DREF
      YMOON   = YONE*STHETA*DREF
      R2      = DSQRT( XMOON**2 + YMOON**2 )
      RADM(J) = R2/RMOON
      CTHETM  = XMOON/R2
      STHETM  = YMOON/R2
      THETAM  = DACOS( CTHETM )
      IF( STHETM .LT. 0.D0 ) THETAM = 2.D0*PI - THETAM
      THETM(J) = THETAM*R2D
C
C  CONVERT VELOCITY IN A ROTATING, GEOCENTRIC FRAME TO ROTATING
C  X-Y COORDS.
C
      XDOTE = YTWO*CTHETA - YTHR*STHETA
      YDOTE = YTWO*STHETA + YTHR*CTHETA
C
C  CONVERT VELOCITY IN A ROTATING, GEOCENTRIC X-Y FRAME TO FIXED MOON
C  CENTERED POLAR COORDS.  ( km/s )
C
      VRADM(J)  =  ( XDOTE*CTHETM + YDOTE*STHETM )*VREF
      VTHETM(J) =  ( -XDOTE*STHETM + YDOTE*CTHETM )*VREF
     &                 + WSYN*R2
      IF ( J .LT. IEND2 )   GO TO 2003
C
C  GO TO THIRD INTEGRATION ARC ( LUNAR CAPTURE SPIRAL )
```

```
C
      JPAST = J
      GO TO 333
C
C  STANDARD FOURTH-ORDER RUNGE-KUTTA INTEGRATION CODE FOLLOWS
C  ( FOR SECOND INTEGRATION ARC )
C
 2001 H2 = 0.5D0*DT
      H6 = DT/6.D0
      DO 2002  I=1, NV
        DS(I) = D(I)
 2002   XS(I) = Y(I)
 2003 DO 2004  I=1, NV
 2004   Y(I) = XS(I) + H2*DS(I)
      IN = 2
      GO TO 201
 2005 DO 2006  I=1, NV
        DD = D(I)
        VS(I) = DS(I) + 2.D0*DD
 2006   Y(I) = XS(I) + H2*DD
      IN = 3
      GO TO 204
 2007 DO 2008  I=1, NV
        DD = D(I)
        VS(I) = VS(I) + 2.D0*DD
 2008   Y(I) = XS(I) + DT*DD
      IN = 4
      GO TO 202
 2009 DO 2010  I=1, NV
        VSS = H6*( VS(I) + D(I) )
        XS(I) = XS(I) + VSS
 2010   Y(I) = XS(I)
      IN = 5
      GO TO 204
 2011 DO 2012  I=1, NV
 2012   DS(I) = D(I)
      GO TO 205
C
  333 CONTINUE
C
```

```
C ****************************************************
C ************   THIRD INTEGRATION SEGMENT  ***************
C ************  LUNAR CAPTURE SPIRAL, ENGINE ON  **********
C ****************************************************
C
C
C  ***  STATE VECTOR I.C. (ROTATING LUNAR POLAR COORDS)  ***
C
      Y(1) = RADM(JPAST)*RMOON/DREF
      Y(2) = VRADM(JPAST)/VREF
      Y(3) = ( VTHETM(JPAST) - WSYN*RADM(JPAST)*RMOON )/VREF
      Y(4) = THETM(JPAST)*D2R
C
C  CALCULATE MOON REL. ENERGY ( km**2/s**2 )
C
      VRELM = DSQRT( VRADM(JPAST)**2 + VTHETM(JPAST)**2 )
      ENERGM(JPAST) = VRELM**2/2.D0 - GMM/(RADM(JPAST)*RMOON)
C
C  CALCULATE MOON REL. ANGULAR MOMENTUM AND ECCENTRICITY
C
      ANGMM  = RADM(JPAST)*RMOON*VTHETM(JPAST)
      ECCENM(JPAST) = DSQRT(1.D0+(2.D0*ENERGM(JPAST)*ANGMM**2)/
     &                  (GMM**2))
C
C  SET TIMELINES FOR LATER CONTROL HISTORY INTERPOLATIONS
C  ( MOON CAPTURE SPIRAL )
C
      NXU  = 11
      DO 41 I=1,NXU
   41 XUM(I) = X(I+9)
      XIN2(1) = 0.D0
      XIN2(NXU) = TF - TON
      NXUM1 = NXU - 1
      DELT = ( TF - TON )/DFLOAT(NXUM1)
      DO 51  I=2, NXUM1
   51 XIN2(I) = XIN2(I-1) + DELT
      XOUT2(1) = 0.D0
      XOUT2(IEND3D) = TF - TON
      HDT = 0.5D0*DT3
      DO 61  I=2, NINT3D
```

```
 61   XOUT2(I) = XOUT2(I-1) + HDT
C
C INTERPOLATE AMONG THE CONTROL POINTS TO OBTAIN A CONTROL TIME
C HISTORY SUITABLE FOR THE NUMERICAL INTEGRATION WHICH FOLLOWS.
C THERE ARE TWICE AS MANY CONTROL ENTRIES AS STATE ENTRIES TO
C ACCOMMODATE THE MIDPOINT EVALUATIONS REQUIRED BY THE RUNGE-KUTTA
C INTEGRATION.
C
      CALL SPLINE(NXU,XIN2,XUM,IEND3D,XOUT2,U,IERR)
C
C  RE-DEFINE THE INITAL MASS (MASSIC, kg) USING THE PRIOR
C  MASS VALUE FROM THE SECOND ARC
C
      MASSIC = MASSI
C
C *************************************************************
C INTEGRATE THE STATE EQUATIONS FOR THE CURRENT CONTROL HISTORY.
C *************************************************************
C
      NV = 4
      DT = DT3
      J  = JPAST
      JC = 1
      IN = 1
      CONTRL = U(1)
      UPRNT(J) = CONTRL*R2D
      GO TO 303
  301 JP1  = J + 1
      JCP1 = JC + 1
      TAU  = TAU + DT
      K = JC + JC
      CONTRL = U(K)
      GO TO 303
  302 K = K + 1
      CONTRL = U(K)
  303 CONTINUE
  304 YONE = Y(1)
      YTWO = Y(2)
      YTHR = Y(3)
      YFOR = Y(4)
```

```
C
C  CALCULATE THE DISTANCES FROM THE EARTH TO THE S/C AND
C  THE MOON TO THE S/C
C
C
C  CALCULATE THE ANGLE FROM THE ROTATING X-AXIS TO THE R2
C  VECTOR
C
      CTHETA = DCOS(YFOR)
      STHETA = DSIN(YFOR)
      R2 = YONE
      XM = R2*CTHETA
      YM = R2*STHETA
      R1 = DSQRT( (1.DO - XM)**2 + YM**2 )
C
C  UPDATE S/C MASS
C
      MASSI = MASSIC - MDOT*( TAU - TON )*TREF
      ATHRUS = (THRUST/MASSI)/AREF
C
C  CALCULATE THE THRUST ACCELERATION IN km/s**2 AND CONVERT
C  TO DIMENSIONLESS UNITS OF DU/TU**2
C
C
C  RADIAL & CIRCUMFERENTIAL THRUST ACCELERATIONS
C
      ATRAD  = ATHRUS * DSIN(CONTRL)
      ATTHET = ATHRUS * DCOS(CONTRL)
C
C  CONVERT RADIAL AND CIRCUMFERENTIAL THRUST ACCL'S TO
C  INERTIAL COORD'S
C
      ATX = ATRAD*CTHETA - ATTHET*STHETA
      ATY = ATRAD*STHETA + ATTHET*CTHETA
C
C  CALCULATE THE ACCEL'N DUE TO THE POTENTIAL FUNCTION (GRAVITY
C  FIELD) IN THE INERTIAL COORD'S
C
      OMEGAX = -( ( MU1*(XM - 1.DO)/(R1**3) ) +
     &            ( MU2*XM/(R2**3) ) ) - MU1
```

```
      OMEGAY = -((MU1*YM/(R1**3)) + (MU2*YM/(R2**3)))
C
C  CALCULATE X AND Y DOT AND DDOT
C
      XDM    = YTWO*CTHETA - YTHR*STHETA
      YDM    = YTWO*STHETA + YTHR*CTHETA
      XDD    = OMEGAX + ATX + 2.DO*YDM + XM
      YDD    = OMEGAY + ATY - 2.DO*XDM + YM
C
C  CONVERT TO POLAR COORDS
C
      ARAD  =  XDD*CTHETA + YDD*STHETA
      ATHET = -XDD*STHETA + YDD*CTHETA
C
C  DIFF. EQ'S OF MOTION ...
C
      D(1) = YTWO
      D(2) = ARAD + YTHR**2/YONE
      D(3) = ATHET - YTWO*YTHR/YONE
      D(4) = YTHR/YONE
      GO TO (3001,3005,3007,3009,3011), IN
  305 J  = JP1
      JC = JCP1
      Y1(J) = YONE
      Y2(J) = YTWO
      Y3(J) = YTHR
      Y4(J) = YFOR
      RADM(J)    = YONE*DREF/RMOON
      VRADM(J)   = YTWO*VREF
      VTHETM(J) = ( YTHR + YONE )*VREF
      THETM(J)   = YFOR*R2D
      TAUS(J)    = TAU*TREF
      MASS(J)    = MASSI
      UPRNT(J)   = CONTRL*R2D
C
C  CALCULATE MOON REL. ENERGY ( km**2/s**2 )
C
      VRELM = DSQRT( VRADM(J)**2 + VTHETM(J)**2 )
      ENERGM(J) = VRELM**2/2.DO - GMM/(RADM(J)*RMOON)
C
```

```
C  CALCULATE MOON REL. ANGULAR MOMENTUM AND ECCENTRICITY
C
      ANGMM   = RADM(J)*RMOON*VTHETM(J)
      ECCENM(J) = DSQRT(1.DO+(2.DO*ENERGM(J)*ANGMM**2)/(GMM**2))
      IF ( J .LT. IEND )   GO TO 3003
C
C  EVALUATE THE PERFORMANCE INDEX ( ENGINE-ON TIME )
C
      TESC    = TOFF
      TAUF    = TAUS(IEND)/TREF
      TCAPT   = TAUF - TON
      F(1)    = TESC + TCAPT
      FMIN    = F(1)
      MASFIN = MASSO - MDOT*(TESC + TCAPT)*TREF
C
C  CALCULATE THE REMAINING PARAMETERS
C
      SMAX    = -GMM/( 2.DO*ENERGM(IEND) )
      SEMPAR = SMAX*( 1.DO - ECCENM(IEND)**2 )
      HPMOON = SEMPAR/( 1 + ECCENM(IEND) ) - RMOON
      HAMOON = SEMPAR/( 1 - ECCENM(IEND) ) - RMOON
C
C  EVALUATE THE CONSTRAINTS :
C
C   THE S/C MUST TERMINATE IN A CIRCULAR ORBIT WITH
C   AN ALTITUDE OF 100 km ABOVE THE MOON
C
      RPARK   = (RMOON + ALTLUN)/RMOON
      VPARK   = DSQRT( GMM/(ALTLUN + RMOON) )
      H(1)    = RPARK - RADM(IEND)
      H(2)   = VRADM(IEND)
      H(3)   = DABS( VTHETM(IEND) ) - VPARK
C
C  PRINT THE PERFORMANCE INDEX VALUE AND THE CONSTRAINTS.
C
      IF (IPRINT .GE. 5) THEN
          WRITE(IOUT,9001) F(1)
          IF (NH .NE. 0) WRITE(IOUT,9002) (H(I),I=1,NH)
          IF (NG .NE. 0) WRITE(IOUT,9003) (G(I),I=1,NG)
      ELSE
```

```
          IF (IGO .EQ. 7) THEN
              IF (IPRINT .GE. 3 .AND. IPRINT .LE. 4) THEN
                  WRITE(IOUT,9001) F(1)
                  IF (NH .NE. 0) WRITE(IOUT,9002) (H(I),I=1,NH)
                  IF (NG .NE. 0) WRITE(IOUT,9003) (G(I),I=1,NG)
              END IF
          END IF
      END IF
      RETURN
C
C  STANDARD FOURTH-ORDER RUNGE-KUTTA INTEGRATION CODE FOLLOWS
C  ( FOR THE THIRD INTEGRATION ARC )
C
 3001 H2 = 0.5D0*DT
      H6 = DT/6.D0
      DO 3002  I=1, NV
        DS(I) = D(I)
 3002   XS(I) = Y(I)
 3003 DO 3004  I=1, NV
 3004   Y(I) = XS(I) + H2*DS(I)
      IN = 2
      GO TO 301
 3005 DO 3006  I=1, NV
        DD = D(I)
        VS(I) = DS(I) + 2.D0*DD
 3006   Y(I) = XS(I) + H2*DD
      IN = 3
      GO TO 304
 3007 DO 3008  I=1, NV
        DD = D(I)
        VS(I) = VS(I) + 2.D0*DD
 3008   Y(I) = XS(I) + DT*DD
      IN = 4
      GO TO 302
 3009 DO 3010  I=1, NV
        VSS = H6*( VS(I) + D(I) )
        XS(I) = XS(I) + VSS
 3010   Y(I) = XS(I)
      IN = 5
      GO TO 304
```

```
 3011 DO 3012  I=1, NV
 3012   DS(I) = D(I)
      GO TO 305
C
C FORMAT STATEMENTS ARE COLLECTED BELOW.
C
 9000 FORMAT('0',6X,'U(CONT. VAR) =',5(1X,I3,1X,D15.8)/,1X,
     >     40(6(1X,I3,1X,D15.8)/,1X))
 9001 FORMAT(1X,'OBJ. FUNCTION =',2X,D16.8)
 9002 FORMAT(1X,'EQUALITIES =',5X,6(D15.8,2X)/,
     >     15(17X,6(D15.8,2X)/))
 9003 FORMAT(1X,'INEQUALITIES =',3X,6(D15.8,2X)/,
     >     15(17X,6(D15.8,2X)/))
      END
```