Market-based pricing strategies for lot-size producers and electric power suppliers

by

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A Thesis Submitted to the

Graduate Faculty in Partial Fulfillment of the

Requirements for the Degree of

MASTER OF SCIENCE

Departments: Industrial and Manufacturing Systems Engineering
Major: Industrial Engineering

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niversity
Iowa

1993

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ACKNOWLEDGEMENT

I would like to express my sincere appreciation and thanks to my major professor, **Dr. K. Jo Min**, for his advice, guidance, and patience throughout all phases of my graduate study and the preparation of this thesis.

I am also grateful to **Dr. H. T. David** and **Dr. Robert Stephenson** for their comments and insightful suggestions regarding my research as well as their participation on my graduate committee.

Finally, I would like to express my warmest thanks to my family for their many years of support, love, and tremendous encouragement. In particular, I would like to thank my **Mom, Ru-Yale Lin**, my Dad, **Teng-Po Chen**, and my sisters.

GENERAL INTRODUCTION

General Background and Objective

In this thesis, we study how lot-size producers and electric power suppliers determine optimal prices and other critical economic quantities (e.g., the order quantities for the lot-size producers and the allocation priorities for the electric power suppliers). Recently, the traditional economic order quantity model has been extended to the case of monopolistic and oligopolistic lot-size producers under profit maximization (see e.g., Min (1992a)). In this thesis, we further extend the general framework of the monopolistic and oligopolistic lot-size models by considering various aspects of model environments (e.g., competitive behavioral assumptions (Cournot vs. Bertrand), reduction of setup and inventory holding costs, purchasing and sales strategies, and performance criteria (profit maximization vs. ROII (return on inventory investment) maximization). On the other hand, for electric power suppliers, we formulate an expected total surplus (i.e., profit plus customers' net benefits) maximization model as a nonlinear programming problem when the amount of electric power demanded and its valuation to customers as well as the amount of electric power supplied are random. In addition, under the assumption that customers are risk-averse, we formulate an interruption insurance model to transfer the risk of customers to the risk-neutral electric power supplier. The effects of errors due to the assumptions that customers' valuation and/or the amount of electric power demanded are constant over time are investigated via numerical examples. A brief introduction of background and motivation for our study (first for the lot-size producers, then for the electric power suppliers) is as follows.

Keeping an inventory to meet potential demand in the future is prevalent in most businesses. Manufacturers, wholesalers, and retailers general have a stock of goods on hand. How to determine the "inventory policies" (i.e., when and how much to produce as well we how much to charge per unit) becomes a critical issue for lot-size producers. A simple model representing production-inventory situation is given by the well-known traditional economic order quantity (EOQ) model (see e.g, Hillier and Lieberman (1990)).

The traditional EOQ model formulates the production-inventory system by considering only cost factors consisting of a fixed setup cost, a variable unit production cost, and an inventory holding cost. It should be pointed out, however, that the inventory policies of numerous businesses may depend on its relations to other business policies regarding pricing and sales. In this thesis, we attempt to integrate the policies of inventory and pricing/sales so as to maximize the policy maker's profit. In a recent paper by Min (1992a), it is assumed that the demand of customers depends on the price a lot-size producer charges and a profit maximizing model of inventory and quantity discount pricing policies for a monopolist is presented. Also, Min (1992b) extended the profit maximizing model to the case of a symmetric oligopoly, under Cournot behavioral assumptions, consisting of lot-size producers of a single homogeneous product who compete with each other for the same potential buyers. In this thesis, we extend the general frameworks of Min (1992a, 1992b) to different environments. First, we compare and contrast the economic implications of equilibria under Cournot and Bertrand behavioral assumptions and perform sensitivity analysis on the decision variables such as market price and order quantity with respect to the parameters such as number of competing lot-size producers and the levels of setup and inventory holding costs. This competitive inventory and pricing model forms the basis for an economic decision model of setup cost and inventory holding cost reductions. The setup and inventory holding cost reductions model demonstrates that the competition among lot-size producers induces setup and inventory holding cost reductions. Also, by incorporating the special structure concerning the purchasing and sales activities of intermediary firms and by modifying the traditional EOQ model accordingly, we will show how to formulate the profit maximization problem for the intermediary firms. Finally, for a single seller, we compare and contrast the optimal inventory and pricing policies under profit maximization vs. ROII (return on inventory investment) maximization when demand is linear in price. By studying the optimality conditions and the corresponding closed-form optimal solutions, several interesting economic implications are derived.

For an electric power supplier, we assume that the electric power supply is stochastic and the objective of supplier is to maximize the total surplus (i.e., profit plus costomers' net benefits). A critical issue of such an electric power supplier is how to allocate the scarce electric power in case of potential shortages. In our model, we employ an allocation scheme called the priority rationing which allocates the scarce power to the higher valued consumption units via pricing of the allocation priorities. Moreover, we improve this allocation scheme by incorporating the commonly shared random factors into the customers' valuation of electric power and the estimation uncertainty into the total amount of electric power demanded. In addition, under the assumption that customers are risk-averse, we formulate an interruption insurance model to transfer the risk of customers to the risk-neutral electric power supplier.

An Explanation of the Thesis Organization

This thesis is composed of five papers which may be suitable for publication. In particular, the third paper "**OPTIMAL SELLING QUANTITY AND PURCHASING PRICE FOR INTERMEDIARY FIRMS**" appears in *International Journal of Operations and Production Management* volume 11, number 10, page 64-68, 1991. Some portions of the fifth paper "**PRIORITY RATIONING/PRICING OF ELECTRIC POWER UNDER CUSTOMERS' VALUATION UNCERTAINTY**" appears in *Twenty-ninth Annual Power Affiliate Report* section 23, page 279-289, Electric Power Research Center, Iowa State

University, May, 1992.

In a recent paper by Min (1992b), he introduced a competitive EOQ profit maximizing model under Cournot behavioral assumption. In contrast to Cournot behavioral assumption, in the first paper "**ECONOMIC ORDER QUANTITY (EOQ) MODELS UNDER COMPETITION WITH SENSITIVITY ANALYSIS**", we present an alternative behavioral assumption called Bertrand behavioral assumption (see e.g., Friedman (1990)). By examining the equilibrium conditions and subsequent sensitivity analyses under these two assumptions, we derive economic relations of critical elements of EOQ models (such as order quantities per cycle) as well as critical elements of the microeconomic market theory (such as market prices).

In paper 2 "**A COMPETITIVE EOQ MODEL WITH OPTIONS TO REDUCE SETUP AND INVENTORY HOLDING COSTS**", the basic model environments (such as setup and per unit production costs as well as customer demand functions) and the assumptions on the model environments are analogous to Cournot model in Min (1992b) with the exception that we assume the options of investing in reducing the setup and inventory holding costs are available. By examining the economic implications in equilibrium and the subsequent sensitivity analysis, we present a unique insight (cf. Porteus (1985) and Zangwill (1987))as to why several Japanese and American producers are striving to reduce the setup costs under ever increasing competition. Specifically, it will be shown that, for a profit maximizing producer, as the number of competing producers increases, his optimal strategy dictates that he reduce his setup and inventory holding costs.

In paper 3 "**OPTIMAL SELLING QUANTITY AND PURCHASING PRICE FOR INTERMEDIARY FIRMS**", how intermediary firms can optimally determine both selling quantity and purchasing price of a product is investigated. By incorporating the special structure of intermediary firms' environments and by modifying the conventional economic order quantity (EOQ) model accordingly, we provide optimal decision rules regarding the selling quantity and purchasing price for intermediary firms.

In paper 4 "**AN ANALYSIS OF OPTIMAL INVENTORY AND PRICING POLICIES UNDER LINEAR DEMAND**", for a single seller, we compare and contrast the optimal inventory and pricing policies under profit maximization vs. ROII (return on inventory investment, see e.g., Rosenberg(1990)) maximization when demand is linear in price. By studying the optimality conditions and the corresponding closed-form optimal solutions, several interesting economic implications are derived. In particular, we show that when a cost factor such as the setup cost, inventory holding cost per unit per unit time, or per unit ordering cost after the setup is sufficiently high, the choice of the objective between profit maximization and ROII maximization is inconsequential to the seller in so far as his optimal decisions are concerned.

In paper 5 "**PRIORITY PRICING AND INTERRUPTION INSURANCE OF ELECTRIC POWER UNDER CUSTOMERS' VALUATION UNCERTAINTY**", we extend the existing work (see e.g., Chao et al. (1986), Chao et al. (1987), and Wilson(1989)) on the priority rationing of electric power by incorporating commonly shared random factors (such as temperature or humidity) associated with customers' valuation of electric power and the uncertainty associated with the estimation of the total amount of electric power demanded. Next, under the assumption that customers are risk-averse, we formulate an interruption insurance model to transfer the risk of customers to the risk-neutral electric power supplier. Finally, via numerical examples, we attempt to investigate the effects of errors due to the assumptions that customers' valuation and/or the total amount of electric power demanded are constant over time (when they actually vary due to random factors).

The rest of my thesis is organized as follows. First, those five papers mentioned earlier will be presented sequentially. Next, the general conclusion about this thesis is followed by the last paper. Finally, the literature cited in the general introduction and the general conclusion will be made.

PAPER 1.

ECONOMIC ORDER QUANTITY (EOQ) MODEL UNDER COMPETITION WITH SENSITIVITY ANALYSIS

ECONOMIC ORDER QUANTITY (EOQ) MODELS UNDER COMPETITION WITH SENSITIVITY ANALYSIS

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ABSTRACT

We extend the profit maximizing economic order quantity (EOQ) model with a constant demand rate over time to the case of a symmetric oligopoly consisting of sellers of a homogeneous product who compete with each other for the same potential buyers. A key feature differentiating this paper from the extant literature on the economic order quantity (EOQ) is that the competition aspects of the inventory theory are analyzed not only with respect to the number of competing sellers, but also with respect to two strategic behavioral assumptions (called the Cournot and the Bertrand behavioral assumptions) on the sellers regarding their competitors. Under these behavioral assumptions, the formulations and equilibrium strategies of our models explicitly depend on the number of competing sellers. From the resulting equilibrium conditions and subsequent analyses, we derive economic relations of critical elements of EOQ models (such as order quantities per cycle) as well as critical elements of the microeconomic market theory (such as market prices).

1. INTRODUCTION

This paper extends the profit maximizing economic order quantity (EOQ) model with a constant demand rate over time to the case of a symmetric oligopoly consisting of sellers of a single homogeneous product who compete with each other for the same potential buyers. The primary goal of this study is to understand economic implications of the resulting equilibrium in terms of critical elements of EOQ models (such as the sales quantity per unit time, the order quantity per cycle, the production (or order) cost and inventory holding cost) as well as critical elements of the microeconomic market theory (such as the market price, the demand elasticity of buyers, and the number of competing sellers).

Specifically, we will derive and compare the sellers' decision variables such as optimal economic order quantities, sales quantities per unit time, and the market prices in equilibria under a Cournot-like behavioral assumption (i.e., each seller first predicts his competitors' sales quantities per unit time in maximizing his own profit; see e.g., Oren, Smith and Wilson [16]) and under a Bertrand-like behavioral assumption (i.e., each seller first predicts his competitors' per unit price in maximizing his own profit; see e.g., Friedman [5]). Furthermore, via sensitivity analyses, we derive and compare the directions and magnitudes of changes in the aforementioned decision variables with respect to changes in inventory holding cost, setup cost and the number of competing sellers under both Cournot-like and Bertrand-like behavioral assumptions.

The idea of employing profits as a performance measure of EOQ type models has been explored as early as the 1950's (see, e.g., Whitin [24] or Smith [19]). Ladany and Sternlieb [10] not only uses the profit levels as the performance measure, but also provides insights on relations among price, cost, and demand by making the demand dependent on the price and the price dependent on the cost and a fixed mark-up. Brahmbhatt and Jaiswal [2] extends the previous model by incorporating variable mark-up as a function of a capital intensity measure and by maximizing profit over the order quantity and the capital intensity. Arcelus and Srinivasan [1] also extends Ladany and Sternlieb [10] by treating demand as a function of price, price as a function of a variable mark-up rate times a unit cost under profit maximization over the order quantity and the variable mark-up rate. Moreover, Monahan [15] as well as Lal and Staelin [11] developed quantity discount schemes for the seller. Lee and Rosenblatt [12] extended Monahan [15] by incorporating more realistic features (e.g., constraints imposed on the amount of discount that can be offered). The rationale for the quantity discount in these papers is the cost savings resulting from coordination of sellers' production quantities and buyers' order quantities under the assumption that both buyers as well as sellers are EOQ based decision makers. The assumption that buyers are EOQ based decision makers is relaxed in a new quantity discount EOQ model in Min [13]. In Min [13], the rationale for the quantity discount is the seller's exploitation of the heterogeneous preferences of the buyers regarding their purchase sizes. More recently, in Min [14], for both uniform pricing and quantity discount pricing (under the heterogeneous buyers' preferences assumption) cases, how to incorporate competition aspects of sellers into EOQ models based on Cournot-like behavioral assumption is discussed.

Also, under the assumption of dynamic and deterministic demands, there have been numerous studies investigating the optimal relations of production schedules, prices, and inventories (see e.g., Gaimon [6], Pekelman [17], Kunreuther and Schrage [9], and Thomas [21]) Thomas [22] investigates the optimal relations of production quantities and prices under the assumption of stochastic demands. Moreover, in Gaimon [7], the assumption of a single firm is replaced by a duopoly, and the optimal relations between production capacities and prices are studied within a differential game framework. Also, in Dockner and Jorgensen [4], optimal pricing strategy under competition is examined and non-cooperative as well as cooperative equilibria results are obtained. In Teng and Thompson [20], an oligopoly model is analyzed and optimal advertising policies are obtained when production costs obey a learning curve. We note that the models constructed and analyzed in the last three papers are also time dependent dynamic models.

In this paper, we will refer to the model under the Cournot-like behavioral assumption as the Cournot model while the model under the Bertrand-like behavioral assumption as the Bertrand model. For both models, we assume that all critical economic quantities sellers must determine such as the optimal economic order quantity and price schedule are made under the framework of static decision making (cf. dynamic decision making framework; see e.g., [6], [17], [9], and [21]). In order to highlight the optimal relations among the critical economic quantities that are derived under the static decision making framework, we will make the following assumptions. We assume: 1) the demand is deterministic and constant over time; 2) production occurs (or orders arrive) instantaneously; 3) there is no learning effects in setup or production. Also we will not consider discounting prices and costs over time and other time dependent features such as promotion and advertising. In addition, we will assume that each seller can produce (or order) sufficient amount of products to meet any quantity demanded by buyers. Under these assumptions, we formulate Cournot and Bertrand models consisting of a systematic oligopoly of n sellers (i.e., sellers are identical in all economic respects such as production costs) offering a homogeneous product. From these formulations, we obtain symmetric Cournot and Bertrand equilibria. For both Cournot and Bertrand models, the formulations and equilibrium strategies explicitly depend on n, the number of competing sellers in the market. In equilibria, we derive interesting economic implications regarding prices, demand elasticities, the number of competitors, average and marginal production costs and average inventory holding costs.

The rest of this paper is organized as follows. In section 2 and 3, we construct and analyze the Cournot model and the Bertrand model respectively. In sections 4 and 5, we perform the sensitivity analyses on the Cournot model and the Bertrand model respectively. In addition, in section 5, we compare and contrast the results from the equilibria and sensitivity analyses of the Cournot and Bertrand models. Finally, in section 6, we summarize and conclude.

2. THE COURNOT MODEL AND ECONOMIC IMPLICATIONS IN EQUILIBRIUM

For the construction of the Cournot Model, we will closely follow Min [14]. We assume that there are n identical sellers (producers or distributors) offering a single product. Also we assume that buyers have perfect information about the per unit prices n sellers charge. Hence, in equilibrium, all sellers will charge the same per unit price, p, the market price. For each seller i, $i = 1, \cdots, n$, as in the cases of traditional EOQ models (see e.g., Hillier and Lieberman [8]), we assume: 1) the goods are produced (or ordered) in equal numbers, Q_i at a time; 2) all Q_i units arrive without delivery lag; 3) no shortage to a buyer is permitted. We also assume that, for each seller i, the total cost per cycle consists of a production (or order) cost and an inventory holding cost. The production (or order) cost per cycle is represented by $K + C(Q_i)$ where K is the setup cost and $C(Q_i)$ is the production (or order) cost incurred in producing (or ordering) Q_i units after the setup. On the other hand, the inventory holding cost is characterized by h, inventory holding cost per unit per unit time. As implied earlier, K, $C(Q)$ and h are identical for all sellers. We further assume that $C(Q)$ is strictly increasing and convex in Q, i.e., $C'(Q) > 0$ and $C''(Q) \geq 0$.

The sales quantity (to buyers or customers) per unit time for the entire market is characterized by $d(p)$, a function of per unit market price p. We assume that the sales quantity, given a price, is constant over time. Also we assume that the sales quantity function is strictly decreasing in p, i.e., $d'(p) < 0$. Under the monotonicity assumption of $d'(p) < 0$, the inverse function $p(d)$ exists (with $p'(d) < 0$). The inverse function $p(d)$ specifies the price p that clears d units in the market. We will assume that the inverse function $p(d)$ is concave in d, i.e., $p''(d) \leq 0$. Just as in microeconomic theory (see e.g., Varian [23]), we can refer to $p(d)$ as the inverse demand function while $d(p)$ as the demand function. Since the demand function $d(p)$ is assumed to be constant over time, so is the inverse demand function $p(d)$.

Given the above definitions and assumptions, we develop a Cournot-like framework as follows. We assume that each seller i, $i = 1, \cdots, n$ will predict the total sales quantity per unit time of his n-1 competitors, d_{-i}. Under this prediction, seller i maximizes his profit per unit time over his sales quantity per unit time d_i and economic order quantity Q_i. For the total sales quantity per unit time for the entire market, $d_i + d_{-i}$, the corresponding per unit market price is given by $p(d_i + d_{-i})$. Hence, the total revenue per cycle for seller i is $p(d_i + d_{-i})\, Q_i$. And the corresponding total cost per cycle and the cycle length are given by $K + C(Q_i) + hQ_i^2/(2d_i)$ and Q_i/d_i respectively. Given these expressions for the total revenue, cost, and the cycle length, the problem of maximizing profit per unit time for seller i, π_i, can be stated as follows.

$$\max_{d_i, Q_i} \ \pi_i = p(d_i + d_{-i})d_i - (K + C(Q_i))d_i/Q_i - hQ_i/2 \tag{1}$$

The first order optimality conditions of the maximization problem (1) are:

$$\frac{\partial \pi_i}{\partial d_i} = p'(d_i + d_{-i})d_i + p(d_i + d_{-i}) - (K + C(Q_i))/Q_i = 0 \qquad (2)$$

$$\frac{\partial \pi_i}{\partial Q_i} = -d_i(C'(Q_i)Q_i - K - C(Q_i))/Q_i^2 - h/2 = 0 \qquad (3)$$

In order to derive the corresponding second order sufficient condition(s) for optimality, we first obtain the second order derivatives of the profit as follows.

$$\frac{\partial^2 \pi_i}{\partial d_i^2} = p''(d_i + d_{-i})d_i + 2p'(d_i + d_{-i}) \qquad (4)$$

$$\frac{\partial^2 \pi_i}{\partial d_i \partial Q_i} = (K + C(Q_i) - Q_i C'(Q_i))/Q_i^2 \qquad (5)$$

$$\frac{\partial^2 \pi_i}{\partial Q_i^2} = d_i(- Q_i^2 C''(Q_i) - 2(K + C(Q_i) - Q_i C'(Q_i)))/Q_i^3 \qquad (6)$$

From our assumptions that $p'(\cdot) < 0$ and $p''(\cdot) \leq 0$, we have $\frac{\partial^2 \pi_i}{\partial d_i^2} < 0$. Therefore, the second order sufficient condition for optimality is simply

$$\frac{\partial^2 \pi_i}{\partial d_i^2} \frac{\partial^2 \pi_i}{\partial Q_i^2} - \left[\frac{\partial^2 \pi_i}{\partial d_i \partial Q_i} \right]^2 > 0 \tag{7}$$

where $\frac{\partial^2 \pi_i}{\partial d_i^2}$, $\frac{\partial^2 \pi_i}{\partial d_i \partial Q_i}$, and $\frac{\partial^2 \pi_i}{\partial Q_i^2}$ are given by (4), (5) and (6).

Throughout the rest of this paper, we will assume that the second order sufficient condition is satisfied for the region of interest. In addition, we will assume that the resulting profit level of each seller i, $i = 1, \cdots, n$, evaluated at the optimal sales quantity per unit time and order quantity per cycle is strictly positive (i.e., no seller will exit from the market).

So far we have examined the optimality conditions of a single seller. We now proceed to derive an equilibrium of n sellers. Under our assumption of identical sellers, there exists a symmetric equilibrium (see e.g., Oren, Smith, and Wilson [16]) where

$$d_1 = d_2 = \cdots = d_n \tag{8}$$

and

$$Q_1 = Q_2 = \cdots = Q_n \tag{9}$$

i.e., the sales quantity per unit time as well as the economic order quantity are identical for all sellers. In this symmetric equilibrium, the total sales quantity per unit time from all competitors of seller i, d_{-i} is equal to $(n-1)d_i$ for $i = 1, \cdots, n$. Therefore, the corresponding equilibrium conditions of the optimality conditions (2) and (3) are given by

$$p'(nd_i)d_i + p(nd_i) - (K + C(Q_i))/Q_i = 0 \tag{10}$$

$$-d_i(C'(Q_i)Q_i - K - C(Q_i))/Q_i^2 - h/2 = 0 \tag{11}$$

Let us first examine equilibrium condition (10). The corresponding demand elasticity ϵ, $\epsilon = p(d)/(p'(d)d)$ by definition (see e.g., Varian [23]), evaluated at the symmetric equilibrium point becomes:

$$\epsilon = p(nd_i)/(nd_i p'(nd_i)) \tag{12}$$

Hence, in the symmetric equilibrium, equation (10) can be restated as

$$p(nd_i) = \left(\frac{n\epsilon}{n\epsilon + 1}\right)(K + C(Q_i))/Q_i \tag{13}$$

Equation (13) states that, given a fixed number of competitors n, as the demand becomes more elastic (i.e., $|\epsilon|$ gets larger), the equilibrium price gets closer to the average production cost. Or as the demand gets more inelastic (i.e., $|\epsilon|$ gets smaller), the equilibrium price gets farther away from the average production cost. If we view the term $\frac{n\epsilon}{n\epsilon + 1}$ as a markup rate, the economic implication is that the markup rate is larger when the demand is more inelastic. On the other hand, given a fixed level of elasticity, ϵ, we observe that as the number of competitors increases (i.e., as the competition gets more intense), the price gets closer to the average production cost. Or as the number of competitors decreases (i.e., as the competition gets less intense), the price gets farther away from the average production cost. We also observe that as the number of competitors decreases, the markup rate increases. In addition, we note that if $-1 < n\epsilon < 0$, the price is negative. Furthermore if $n\epsilon = -1$, it can be easily verified that no order quantity per cycle Q_i satisfies equation (10). Hence, throughout this paper, we limit our analysis to the cases where $n\epsilon < -1$. i.e., $n\epsilon < -1$ will be assumed.

Let us now examine equilibrium condition (11). By rearranging terms of condition (11), we have

$$(K + C(Q_i))/Q_i - C'(Q_i) = hQ_i/(2d_i) \qquad (14)$$

Equilibrium condition (14) states that for each seller i, $i = 1, \cdots, n$, the average production cost is equal to the sum of the marginal production cost and the average inventory cost per unit. The economic implication is that the per unit production cost is strictly higher than the per unit inventory cost at the equilibrium since the marginal production cost is assumed to be positive. Also we note that if $(K + C(Q_i))/Q_i \leq C'(Q_i)$, it can be easily verified that no order quantity per cycle Q_i satisfies equation (11). Hence, throughout this paper, we limit our analysis to the cases where $(K + C(Q_i))/Q_i > C'(Q_i)$. i.e., $(K + C(Q_i))/Q_i > C'(Q_i)$ will be assumed.

We note that the relation between the equilibrium sales quantity per unit time d_i and the corresponding economic order quantity Q_i for i

$= 1, \cdots, n$ is implicitly determined by (13) and (14). By simultaneously solving conditions (13) and (14) given $p(\cdot)$, $C(\cdot)$, h, K, and n, we can numerically determine the values of d_i and Q_i.

3. THE BERTRAND MODEL AND ECONOMIC IMPLICATIONS IN EQUILIBRIUM

The basic model environments concerning EOQ based decision making sellers are analogous to those in section 2. Also we assume that the sales quantity function is strictly decreasing in p, i.e., $d'(p) < 0$. In contrast to the Cournot-like framework presented in the previous section, we develop a Bertrand-like framework as follows. Let us denote the per unit price seller i charges by p_i, $i = 1, \cdots, n$. We assume that each seller i, $i = 1, \cdots, n$ will predict his n-1 competitors' prices, p_j, $j = 1, \cdots, n$; $j \neq i$. Under our assumptions that the product is homogeneous and that buyers have perfect information about the per unit prices n sellers charge, the following argument holds. If seller i's price p_i is set such that p_i is strictly higher than the lowest price of his n-1 competitors (i.e., $p_i > \bar{p}_{-i} = \min \{p_j | \ j = 1, \cdots, n; \ j \neq i\}$), then no buyer will purchase from seller i. On the other hand, if seller i's price p_i is set such that p_i is strictly lower than the lowest price of his n-1 competitors (i.e., $p_i < \bar{p}_{-i}$), then no buyer will purchase from any of his competitors. Finally, if seller i's price p_i is set such that p_i is equal to the lowest price of his n-1 competitors (i.e., $p_i = \bar{p}_{-i}$) and there are k sellers with the same minimum price (including seller i), then each of the k sellers will equally share the total sales quantity in the entire market. Therefore, for $i = 1, \cdots, n$, seller i's profit per unit time, π_i, conditioned on his price p_i is shown as follows.

$$\pi_i = p_i d(p_i) - d(p_i)(K + C(Q_i))/Q_i - hQ_i/2 \tag{15}$$

if $p_i < \bar{p}_{-i} = \min \{p_j | \ j = 1, \cdots, n; \ j \neq i\}$.

$$= \bar{p}_{-i} d(\bar{p}_{-i})/k - (d(\bar{p}_{-i})/k)(K + C(Q_i))/Q_i - hQ_i/2 \tag{16}$$

if $p_i = \bar{p}_{-i}$ and there are k sellers with the same minimum price (the sales quantity per unit time of seller i is $d(\bar{p}_{-i})/k$)

$$= 0 \quad \text{(achieved by neither producing nor ordering)} \tag{17}$$

if $p_i > \bar{p}_{-i}$.

Seller i will maximize his profit per unit time given in relations (15)-(17) over his price p_i and order quantity Q_i. In the case of relation (17), since seller i's optimal policy is neither producing nor ordering, no further analysis is warranted for. Hence, throughout the rest of this section, we will concentrate on the analysis of relations (15) and (16). For the analysis, we will assume that profit relations (15) and (16) are non-negative and concave in p_i and Q_i in the region of interests. Under these assumptions, the optimality conditions for p_i and Q_i are:

Either

$$p_i < \bar{p}_{-i} \tag{18}$$

from (15), $\frac{\partial \pi_i}{\partial p_i} = d(p_i) + d'(p_i)(p_i - (K + C(Q_i))/Q_i) = 0$ (19)

$$\frac{\partial \pi_i}{\partial Q_i} = d(p_i)(K + C(Q_i) - Q_i C'(Q_i))/Q_i^2 - h/2 = 0 \tag{20}$$

Or

$$p_i = \bar{p}_{-i} \tag{21}$$

$$\frac{\partial \pi_i}{\partial Q_i} = (d(p_i)/k)(K + C(Q_i) - Q_i C'(Q_i))/Q_i^2 - h/2 = 0 \tag{22}$$

where there are k sellers with the same minimum price $p_i = \bar{p}_{-i}$.

The optimal price p_i and order quantity Q_i are implicitly determined from relations (18)-(20), or relations (21) and (22).

So far we have derived the optimality conditions of a single seller. We now proceed to derive an equilibrium of n sellers. Under our assumption that the product is homogeneous, seller i, i = 1, ⋯, n, can capture the entire market by slightly under-cutting the n-1 competitors' prices. Hence, so long as the current level of profit is positive, each seller will under-cut the n-1 competitors' prices. This incentive to under-cut will vanish only if the current level of profit is zero. Therefore, under our assumption of identical sellers, the following relations hold in an equilibrium.

$$p_1 = p_2 = \cdots = p_n \tag{23}$$

$$p_i d(p_i)/n - (d(p_i)/n)(K + C(Q_i))/Q_i - hQ_i/2 = 0 \tag{24}$$

i.e., $\pi_i = 0$ for $i = 1, \cdots, n$.

The corresponding relations on the order quantity Q_i's are given by,

$$Q_1 = Q_2 = \cdots = Q_n \tag{25}$$

$$(d(p_i)/n)(K + C(Q_i) - Q_iC'(Q_i))/Q_i^2 - h/2 = 0 \tag{26}$$

i.e., $\frac{\partial\pi_i}{\partial Q_i} = 0$ (from relation (22)) for $i = 1, \cdots, n$.

A symmetric equilibrium of n sellers is implicitly determined by equilibrium conditions (23)-(26) while seller i's ($i = 1, \cdots, n$) equilibrium price p_i and order quantity Q_i are implicitly determined by (24) and (26). By simultaneously solving conditions (24) and (26) given $d(\cdot)$, $C(\cdot)$, h, K, and n, we can numerically determine the values of p_i and Q_i. From these values and equilibrium conditions (23) and (25), the complete set of equilibrium prices and order quantities can be numerically determined.

We examine equilibrium condition (24) first. Equilibrium condition (24) can be rearranged to become,

$$p_i = (K + C(Q_i))/Q_i + hQ_in/(2d(p_i)) \tag{27}$$

Condition (27) states that for seller i, $i = 1, \cdots, n$, the price (= per unit revenue) is equal to the sum of the per unit production cost plus the per unit inventory holding cost. cf. conditions (2) and (10) in section 2 under the Cournot-like behavioral assumption where the conditions imply that the marginal revenue with respect to the sales quantity per unit time is equal to the marginal cost with respect to the sales quantity per unit time.

Let us now examine equilibrium condition (26). By rearranging terms of condition (26), we have

$$(K + C(Q_i))/Q_i - C'(Q_i) = hQ_i n/(2d(p_i)) \tag{28}$$

The economic interpretations of equilibrium condition (28) are analogous to those of condition (14) in section 2. That is, for each seller i, i = 1, $\cdots$, n, the per unit production cost is equal to the sum of the marginal production cost and the per unit inventory cost per unit. This implies that the per unit production cost is strictly higher than the per unit inventory cost at the equilibrium since the marginal production cost is assumed to be positive. Also we note that if $(K + C(Q_i))/Q_i \leq C'(Q_i)$, it can be easily verified that no order quantity per cycle Q_i satisfies equation (26). Hence, throughout this section, we limit our analysis to the cases where $(K + C(Q_i))/Q_i > C'(Q_i)$. i.e., $(K + C(Q_i))/Q_i > C'(Q_i)$ will be assumed. Furthermore, from conditions (27) and (28), it can be easily seen that in equilibrium the price (per unit revenue) is strictly greater than the marginal production cost (by two times the per unit inventory holding cost). This result is consistent with that in section 2 under the Cournot behavioral assumption. Finally, we note that since the equilibrium profit level of the Bertrand model is always zero while the equilibrium profit level of the Cournot model may be positive, the Cournot profit level is higher than or equal to the Bertrand profit level. This is consistent with the microeconomic market theory (see e.g., Varian [23]).

4. SENSITIVITY ANALYSIS UNDER COURNOT MODEL

In this section, we investigate the sensitivity of the sales quantity per unit time d_i and order quantity per cycle Q_i in equilibrium with respect to the given parameters of the Cournot model depicted in section 2, the inventory holding cost h, the setup cost K, and the number of competing sellers n. Our analysis of sensitivity will be based on differential calculus (especially the implicit function theorem; see e.g., Chiang [3]), which requires variables (or parameters) to be continuous rather than discrete. Hence it will be necessary to treat the number of competing sellers n ($n \geq 1$), which is hitherto assumed to be an integer, as a continuous variable. We present the justification for treating n as a continuous variable (to the extent possible) by slightly rephrasing a portion of section 3, " Modeling Entry ", in Seade [18] as follows:

We will allow n to be an actual continuous variable (or parameter) on which each economic quantity (e.g., price p) depends differentiably according to the given relations, but we restrict our attention to integer realization of this variable. Then, if we define x as any economic quantity dependent on n (e.g., economic order quantities), its change when one additional seller enters into the market is $\Delta x = x(n+1) - x(n)$. It is clear that (sign Δx) = (sign $x'(n)$) whenever the latter sign does not change in the relevant range [n, n+1]; otherwise the sign of Δx is ambiguous. We will assume away cases where this ambiguity arises and hence work with sign $x'(n)$ directly. It is essentially this single-signedness assumption, which one can check, that underlies the common continuous treatment of discrete variables (or parameters) in problems of the present sort.

At the equilibrium point (d_i, Q_i), by applying the implicit function theorem and by allowing n to be continuous, we obtain the following relations for the magnitudes of changes $\frac{\partial d_i}{\partial h}$, $\frac{\partial Q_i}{\partial h}$, $\frac{\partial d_i}{\partial K}$, $\frac{\partial Q_i}{\partial K}$, $\frac{\partial d_i}{\partial n}$, and $\frac{\partial Q_i}{\partial n}$ with respect to an infinitesimal increase in inventory holding cost h, the setup cost K, and the number of competing sellers n. Let F^1 and F^2 denote the left hand sides of equilibrium conditions (10) and (11) respectively. From the assumption that the second order condition (7) is satisfied, the determinant of the Hessian matrix is positive. It can be easily verified that this implies the determinant of Jacobian of F^1 and F^2 with respect to d_i and Q_i (shown in the left hand sides of (29)-(31)) is also positive, satisfying a condition necessary for applying the implicit function theorem. Finally, for the inverse demand and cost quantities, $p(\cdot)$, $p'(\cdot)$, $p''(\cdot)$, $C(\cdot)$, $C'(\cdot)$, and $C''(\cdot)$, the arguments nd_i and Q_i are suppressed for more comprehensible presentation. Then, we have:

$$\begin{bmatrix} \frac{\partial F^1}{\partial d_i} & \frac{\partial F^1}{\partial Q_i} \\ \frac{\partial F^2}{\partial d_i} & \frac{\partial F^2}{\partial Q_i} \end{bmatrix} \begin{bmatrix} \frac{\partial d_i}{\partial h} \\ \frac{\partial Q_i}{\partial h} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial h} \\ -\frac{\partial F^2}{\partial h} \end{bmatrix} \qquad (29)$$

$$\begin{bmatrix} \frac{\partial F^1}{\partial d_i} & \frac{\partial F^1}{\partial Q_i} \\ \frac{\partial F^2}{\partial d_i} & \frac{\partial F^2}{\partial Q_i} \end{bmatrix} \begin{bmatrix} \frac{\partial d_i}{\partial K} \\ \frac{\partial Q_i}{\partial K} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial K} \\ -\frac{\partial F^2}{\partial K} \end{bmatrix} \tag{30}$$

$$\begin{bmatrix} \frac{\partial F^1}{\partial d_i} & \frac{\partial F^1}{\partial Q_i} \\ \frac{\partial F^2}{\partial d_i} & \frac{\partial F^2}{\partial Q_i} \end{bmatrix} \begin{bmatrix} \frac{\partial d_i}{\partial n} \\ \frac{\partial Q_i}{\partial n} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial n} \\ -\frac{\partial F^2}{\partial n} \end{bmatrix} \tag{31}$$

where $\frac{\partial F^1}{\partial d_i}$, $\frac{\partial F^1}{\partial Q_i}$ $(= \frac{\partial F^2}{\partial d_i})$, and $\frac{\partial F^2}{\partial Q_i}$ are calculated to be $np''d_i + (n+1)p'$, $(K + C - Q_iC')/Q_i^2$, and $d_i(-Q_i^2C'' - 2(K + C - Q_iC'))/Q_i^3$ respectively while $\frac{\partial F^1}{\partial h}$, $\frac{\partial F^2}{\partial h}$, $\frac{\partial F^1}{\partial K}$, $\frac{\partial F^2}{\partial K}$, $\frac{\partial F^1}{\partial n}$, and $\frac{\partial F^2}{\partial n}$ are calculated to be 0, $-1/2$, $-1/Q_i$, d_i/Q_i^2, $((\partial p'/\partial n)d_i+(\partial p/\partial n))$, and 0 respectively.

The inverse of the Jacobian matrix exists since its determinant is nonzero. Hence, we solve the systems of equations (29)-(31) for the magnitudes of changes $\frac{\partial d_i}{\partial h}$, $\frac{\partial Q_i}{\partial h}$, $\frac{\partial d_i}{\partial K}$, $\frac{\partial Q_i}{\partial K}$, $\frac{\partial d_i}{\partial n}$, and $\frac{\partial Q_i}{\partial n}$ as follows.

In the following derivations, the quantity G is defined to be the inverse of the determinant of the Jacobian matrix in the left hand sides of (29)-(31). i.e., $1/G = \frac{\partial F^1}{\partial d_i} \frac{\partial F^2}{\partial Q_i} - \left[\frac{\partial F^1}{\partial Q_i} \right]^2$. Then, we have:

$$\begin{bmatrix} \frac{\partial d_i}{\partial h} \\ \frac{\partial Q_i}{\partial h} \end{bmatrix} = G \begin{bmatrix} \frac{\partial F^2}{\partial Q_i} & -\frac{\partial F^1}{\partial Q_i} \\ -\frac{\partial F^2}{\partial d_i} & \frac{\partial F^1}{\partial d_i} \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \tag{32}$$

$$= G \begin{bmatrix} (Q_iC' - K - C)/(2Q_i^2) \\ (np''d_i + (n+1)p')/2 \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} \frac{\partial d_i}{\partial K} \\ \frac{\partial Q_i}{\partial K} \end{bmatrix} = G \begin{bmatrix} \frac{\partial F^2}{\partial Q_i} & -\frac{\partial F^1}{\partial Q_i} \\ -\frac{\partial F^2}{\partial d_i} & \frac{\partial F^1}{\partial d_i} \end{bmatrix} \begin{bmatrix} 1/Q_i \\ -d_i/Q_i^2 \end{bmatrix} \tag{34}$$

$$= G \begin{bmatrix} d_i(-Q_i^2C'' - (K + C - Q_iC'))/Q_i^4 \\ (Q_iC' - K - C)/Q_i^3 - (np''d_i + (n+1)p')d_i/Q_i^2 \end{bmatrix} \tag{35}$$

$$\begin{bmatrix} \frac{\partial d_i}{\partial n} \\ \frac{\partial Q_i}{\partial n} \end{bmatrix} = G \begin{bmatrix} \frac{\partial F^2}{\partial Q_i} & -\frac{\partial F^1}{\partial Q_i} \\ -\frac{\partial F^2}{\partial d_i} & \frac{\partial F^1}{\partial d_i} \end{bmatrix} \begin{bmatrix} -((\partial p'/\partial n)d_i+(\partial p/\partial n)) \\ 0 \end{bmatrix} \tag{36}$$

$$= G \begin{bmatrix} -\ d_i(-Q_i^2C'' - 2(K + C - Q_iC'))((\partial p'/\partial n)d_i+(\partial p/\partial n)) \\ (K + C - Q_iC')((\partial p'/\partial n)d_i+(\partial p/\partial n))/Q_i^2 \end{bmatrix} \tag{37}$$

The corresponding directions of $\frac{\partial d_i}{\partial h}$, $\frac{\partial Q_i}{\partial h}$, $\frac{\partial d_i}{\partial K}$, $\frac{\partial Q_i}{\partial K}$, $\frac{\partial d_i}{\partial n}$, and $\frac{\partial Q_i}{\partial n}$

are summarized in the following proposition (see Appendix A for the proof).

Proposition 1 : *Assume that the sales quantity per unit time* and *order quantity per cycle* (d_i, Q_i) *satisfy the equilibrium conditions* (10) *and* (11) *and the second order sufficient condition* (7). *Assume further that for the cost and the inverse demand functions*, $C'(Q) > 0$, $C''(Q) \geq 0$, $p'(d) < 0$ *and* $p''(d) \leq 0$. *Moreover, assume that the profit level at* (d_i, Q_i) *is positive.*

Then, 1) $\frac{\partial d_i}{\partial h} < 0$ *and* $\frac{\partial Q_i}{\partial h} < 0$,

2) $\frac{\partial d_i}{\partial K} < 0$ *and* $\frac{\partial Q_i}{\partial K} > 0$,

and 3) $\frac{\partial d_i}{\partial n} < 0$ *and* $\frac{\partial Q_i}{\partial n} < 0$.

The economic implications of Proposition 1 are as follows. In the equilibrium, if the inventory holding cost is increased by a small amount, then the sales quantity per unit time as well as the order quantity per cycle will decrease for seller i, $i = 1, \cdots, n$. Figure 1 depicts the resulting changes in inventory levels over time after a small increase in the inventory holding cost. We note that the change in the frequency of ordering is indeterminate (i.e., the corresponding cycle length may be shorter or longer than before the change). Also, in the equilibrium, if the setup cost is increased by a small amount, then the sales quantity per unit time will decrease while the order quantity per cycle will increase for seller i, $i = 1, \cdots, n$. Figure 2 represents the resulting change in inventory levels over time after a small increase in the setup cost. We note that the change in the frequency of ordering is decreased (i.e., the corresponding cycle length is longer than before the change). Finally, under the aforementioned single-signedness assumption, we conclude that if the number of competing sellers increases by a small number, the sales quantity per unit time as well as the order quantity per cycle will decrease in equilibrium. Figure 3 depicts resulting changes in inventory levels over time after a small increase in the number of competing sellers. We note that the change in the frequency of ordering is indeterminate (i.e., the corresponding cycle length may be longer or shorter.). The sensitivity results shown in the proposition also implies that, insofar as the directions of changes in equilibrium are concerned, the impacts of competition on the equilibrium are analogous to those of inventory holding cost on the equilibrium, but not to those of setup cost on the equilibrium.

5. SENSITIVITY ANALYSIS UNDER BERTRAND MODEL

In this section, we investigate the sensitivity of the price p_i and the order quantity per cycle Q_i in equilibrium with respect to the given parameters of the Bertrand model depicted in section 3, the inventory holding cost h, the setup cost K, and the number of competing sellers n. As discussed in section 4, we will treat the number of competing sellers n ($n \geq 1$) as a continuous variable. (see section 4 for details)

At the equilibrium point (p_i, Q_i), by applying the implicit function theorem and by allowing n to be continuous, we will obtain the following relations for the magnitudes of changes $\frac{\partial p_i}{\partial h}$, $\frac{\partial Q_i}{\partial h}$, $\frac{\partial p_i}{\partial K}$, $\frac{\partial Q_i}{\partial K}$, $\frac{\partial p_i}{\partial n}$, and $\frac{\partial Q_i}{\partial n}$ with respect to an infinitesimal increase in inventory holding cost h, the setup cost K, and the number of competing sellers n as follows.

Let E^1 and E^2 denote the left hand sides of equilibrium conditions (24) and (26) respectively. From the assumption that the profit expressions (15) and (16) are concave in p_i and Q_i, $\frac{\partial \pi_i}{\partial p_i} \geq 0$ in the equilibrium. For the successful application of the implicit function theorem, we will further assume that $\frac{\partial \pi_i}{\partial p_i} > 0$ in the equilibrium (see e.g., Chiang [3]). Under this assumption, it can be easily verified that the determinant of Jacobian of E^1 and E^2 with respect to p_i and Q_i (shown in the left hand sides of (38)-(40)) is strictly negative. Since the determinant of the Jacobian is nonzero, an inverse matrix exists. Finally, for the demand and cost quantities, $d(\cdot)$, $d'(\cdot)$, $C(\cdot)$, $C'(\cdot)$, and $C''(\cdot)$, the arguments p_i and Q_i are suppressed for more comprehensible presentation. Then, we have:

$$\begin{bmatrix} \frac{\partial E^1}{\partial p_i} & \frac{\partial E^1}{\partial Q_i} \\ \frac{\partial E^2}{\partial p_i} & \frac{\partial E^2}{\partial Q_i} \end{bmatrix} \begin{bmatrix} \frac{\partial p_i}{\partial h} \\ \frac{\partial Q_i}{\partial h} \end{bmatrix} = \begin{bmatrix} -\frac{\partial E^1}{\partial h} \\ -\frac{\partial E^2}{\partial h} \end{bmatrix} \tag{38}$$

$$\begin{bmatrix} \frac{\partial E^1}{\partial p_i} & \frac{\partial E^1}{\partial Q_i} \\ \frac{\partial E^2}{\partial p_i} & \frac{\partial E^2}{\partial Q_i} \end{bmatrix} \begin{bmatrix} \frac{\partial p_i}{\partial K} \\ \frac{\partial Q_i}{\partial K} \end{bmatrix} = \begin{bmatrix} -\frac{\partial E^1}{\partial K} \\ -\frac{\partial E^2}{\partial K} \end{bmatrix} \tag{39}$$

$$\begin{bmatrix} \frac{\partial E^1}{\partial p_i} & \frac{\partial E^1}{\partial Q_i} \\ \frac{\partial E^2}{\partial p_i} & \frac{\partial E^2}{\partial Q_i} \end{bmatrix} \begin{bmatrix} \frac{\partial p_i}{\partial n} \\ \frac{\partial Q_i}{\partial n} \end{bmatrix} = \begin{bmatrix} -\frac{\partial E^1}{\partial n} \\ -\frac{\partial E^2}{\partial n} \end{bmatrix} \tag{40}$$

where $\frac{\partial E^1}{\partial p_i}$, $\frac{\partial E^1}{\partial Q_i}$, $\frac{\partial E^2}{\partial p_i}$, and $\frac{\partial E^2}{\partial Q_i}$ are calculated to be $d/n + (d'/n)(p_i - (K + C)/Q_i)$, 0, $(d'/n)(K + C - Q_iC')/Q_i^2$, and $(d/n)(-Q_i^2C'' - 2(K + C - Q_iC'))/Q_i^3$ respectively while $\frac{\partial E^1}{\partial h}$, $\frac{\partial E^2}{\partial h}$, $\frac{\partial E^1}{\partial K}$, $\frac{\partial E^2}{\partial K}$, $\frac{\partial E^1}{\partial n}$, and $\frac{\partial E^2}{\partial n}$ are calculated to be $-Q_i/2$, $-1/2$, $-d/(nQ_i)$, $d/(nQ_i^2)$, $-(d/n^2)(p_i - (K + C)/Q_i)$, and $-(d/n^2)(K + C - Q_iC')/Q_i^2$ respectively.

We now solve the systems of equations (38)-(40) for the magnitudes of changes $\frac{\partial p_i}{\partial h}$, $\frac{\partial Q_i}{\partial h}$, $\frac{\partial p_i}{\partial K}$, $\frac{\partial Q_i}{\partial K}$, $\frac{\partial p_i}{\partial n}$, and $\frac{\partial Q_i}{\partial n}$ as follows. Let H denote the inverse of the determinant of the Jacobian matrix in the left hand sides of (38)-(40). i.e., $1/H = \frac{\partial E^1}{\partial p_i}\frac{\partial E^2}{\partial Q_i} - \frac{\partial E^1}{\partial Q_i}\frac{\partial E^2}{\partial p_i}$. Then, we have:

$$\begin{bmatrix} \frac{\partial p_i}{\partial h} \\ \frac{\partial Q_i}{\partial h} \end{bmatrix} = H \begin{bmatrix} \frac{\partial E^2}{\partial Q_i} & -\frac{\partial E^1}{\partial Q_i} \\ -\frac{\partial E^2}{\partial p_i} & \frac{\partial E^1}{\partial p_i} \end{bmatrix} \begin{bmatrix} Q_i/2 \\ 1/2 \end{bmatrix} \tag{41}$$

$$= H \begin{bmatrix} (d/n)(-Q_i^2 C'' - 2(K + C - Q_i C'))/(2Q_i^2) \\ d/(2n) \end{bmatrix} \tag{42}$$

$$\begin{bmatrix} \frac{\partial p_i}{\partial K} \\ \frac{\partial Q_i}{\partial K} \end{bmatrix} = H \begin{bmatrix} \frac{\partial E^2}{\partial Q_i} & -\frac{\partial E^1}{\partial Q_i} \\ -\frac{\partial E^2}{\partial p_i} & \frac{\partial E^1}{\partial p_i} \end{bmatrix} \begin{bmatrix} d/(nQ_i) \\ -d/(nQ_i^2) \end{bmatrix} \tag{43}$$

$$= H \begin{bmatrix} d^2(-Q_i^2 C'' - 2(K + C - Q_i C'))/(n^2 Q_i^4) \\ -d(d + d'(p_i - C'))/(n^2 Q_i^2) \end{bmatrix} \tag{44}$$

$$\begin{bmatrix} \frac{\partial p_i}{\partial n} \\ \frac{\partial Q_i}{\partial n} \end{bmatrix} = H \begin{bmatrix} \frac{\partial E^2}{\partial Q_i} & -\frac{\partial E^1}{\partial Q_i} \\ -\frac{\partial E^2}{\partial p_i} & \frac{\partial E^1}{\partial p_i} \end{bmatrix} \begin{bmatrix} d(p_i - (K + C)/Q_i)/n^2 \\ d(K + C - Q_i C')/(n^2 Q_i^2) \end{bmatrix} \tag{45}$$

$$= H \begin{bmatrix} d^2(-Q_i^2C'' - 2(K + C - Q_iC'))(p_i - (K + C)/Q_i)/(n^3Q_i^3) \\ d^2(K + C - Q_iC')/(n^3Q_i^2) \end{bmatrix} \quad (46)$$

The corresponding directions of $\frac{\partial p_i}{\partial h}$, $\frac{\partial Q_i}{\partial h}$, $\frac{\partial p_i}{\partial K}$, $\frac{\partial Q_i}{\partial K}$, $\frac{\partial p_i}{\partial n}$, and $\frac{\partial Q_i}{\partial n}$ are summarized in the following proposition (see Appendix B for the proof).

Proposition 2 : *Assume that the per unit price and order quantity per cycle* (p_i, Q_i) *satisfy the equilibrium conditions* (24) *and* (26). *Assume further that for the cost and the demand functions,* $C'(\cdot) > 0$, $C''(\cdot) \geq 0$, $d'(\cdot) < 0$. *Moreover, assume that the profit level is non-negative and* $\frac{\partial \pi_i}{\partial p_i} > 0$ *at* (p_i, Q_i).

Then, 1) $\frac{\partial p_i}{\partial h} > 0$ *and* $\frac{\partial Q_i}{\partial h} < 0$;

2) $\frac{\partial p_i}{\partial K} > 0$ while

$\frac{\partial Q_i}{\partial K} > 0$ *if* $d + d'(p_i - C') > 0$,

$\frac{\partial Q_i}{\partial K} = 0$ *if* $d + d'(p_i - C') = 0$,

and $\frac{\partial Q_i}{\partial K} < 0$ *if* $d + d'(p_i - C') < 0$;

3) $\frac{\partial p_i}{\partial n} > 0$ *and* $\frac{\partial Q_i}{\partial n} < 0$.

The economic implications of Proposition 2 are as follows. In the equilibrium, if the inventory holding cost is increased by a small amount, then the per unit price will increase while the order quantity per cycle will decrease for seller i, i = 1, ⋯, n. The sign of the corresponding change in the sales quantity per unit time for the entire market (i.e., $d(p_i)$) will be negative since $d(p_i)$ is strictly monotone decreasing in p_i. Hence, the sign of the corresponding change in the sales quantity per unit time for seller i (i.e., $d(p_i)/n$) will also be negative. That is, the sales quantity per unit time for seller i will decrease if the inventory holding cost is increased by a small amount. In section 4, under the Cournot behavioral assumption, the sales quantity per unit time for seller i as well as the order quantity per cycle decrease if the inventory holding cost is increased by a small amount. Furthermore, the sign of the corresponding change in the per unit price (i.e., $p(nd_i)$ in section 4 where d_i denotes the sales quantity per unit time for seller i) will be positive since $p(nd_i)$ is strictly monotone decreasing in d_i. That is, the per unit price will increase if the inventory holding cost is increased by a small amount. Therefore, we conclude that the directions of changes with respect to the inventory holding cost are identical for both Bertrand and Cournot models.

Also, in the equilibrium, if the setup cost is increased by a small amount, then the per unit price will increase for seller i, i = 1, ⋯, n. On the other hand, the order quantity per cycle will increase, remain the same, or decrease for seller i, i = 1, ⋯, n, depending upon the conditions (in terms of $d(p_i)$, $d'(p_i)$, p_i, and $C'(Q_i)$) given in the proposition. The sign of the corresponding change in the sales quantity per unit time for the entire market (i.e., $d(p_i)$) will be negative since $d(p_i)$ is strictly monotone decreasing in p_i. Hence, the sign of the corresponding change in the sales quantity per unit time for seller i (i.e., $d(p_i)/n$) will also be negative. That is, the sales quantity per unit time for seller i will decrease if the setup cost is increased by a small amount. In section 4, under the Cournot behavioral assumption, sales quantity per unit time for seller i will decrease while the order quantity per cycle will increase if the setup cost is increased by a small amount. Furthermore, the sign of the corresponding change in the per unit price (i.e., $p(nd_i)$ in section 4) will be positive since $p(nd_i)$ is strictly monotone decreasing in d_i. That is, the per unit price will increase if the setup cost is increased by a small amount. Therefore, we conclude that the directions of changes in the per unit price and the sales quantity per unit time for seller i with respect to the setup cost are identical for both Bertrand and Cournot models while the direction of change in the order quantity per cycle for the Bertrand model may be different from the direction of change for the Cournot model.

Finally, under the aforementioned single-signedness assumption, we conclude that if the number of competing sellers increases by a small number, the per unit price will increase while the order quantity per cycle will decrease in equilibrium for seller $i = 1, \cdots, n$. The sign of the corresponding change in the sales quantity per unit time for the entire market (i.e., $d(p_i)$) will be negative since $d(p_i)$ is strictly monotone decreasing in p_i. Let us denote the new sales quantity per unit time for the entire market by $\hat{d}$ ($\hat{d} < d(p_i)$) and the new number of competing sellers by $\hat{n}$ ($\hat{n} > n$). Then, the corresponding new sales quantity per unit time for seller i is given by $\frac{\hat{d}(p_i)}{\hat{n}}$. It can be easily seen that $\frac{\hat{d}(p_i)}{\hat{n}} < \frac{d(p_i)}{n}$. That is, the sales quantity per unit time for seller i will decrease if the number of competing sellers is increased by a small number. In section 4, under the Cournot behavioral assumption, sales quantity per unit time for seller i as well as the order quantity per cycle decrease if the number of competing sellers is increased by a small number. The sign of the corresponding change in the per unit price (i.e., $p(nd_i)$ in section 4), however, is indeterminate due to the following reason. Let $\hat{d}_i$ ($\hat{d}_i < d_i$) denote the new sales quantity per unit time for seller i and let $\hat{n}$ ($\hat{n} > n$) denote the new number of competing sellers. Then, the corresponding new per unit price is given by $\hat{p} = p(\hat{n}\hat{d}_i)$. Since $\hat{n} > n$ and $\hat{d}_i < d_i$, the sign of $p(\hat{n}\hat{d}_i) - p(nd_i)$ is indeterminate. i.e., the per unit price may increase, remain the same, or decrease when the number of competing sellers increase by a small number. Therefore, we conclude that the directions of changes in the sales quantity per unit time for seller i and the order quantity per cycle with respect to the setup cost are identical for both Bertrand and Cournot models while the direction of change in the per unit price for the Bertrand model may be different from the direction of change for the Cournot model.

The sensitivity results shown in the proposition imply that, insofar as the directions of changes in equilibrium are concerned, the impacts of competition on the equilibrium are analogous to those of inventory holding cost on the equilibrium, but may not be analogous to those of setup cost on the equilibrium. We further note that the directions of changes except those of $\frac{\partial Q_i}{\partial K}$ and $\frac{\partial p_i}{\partial n}$ are identical for both Bertrand and Cournot models. i.e., the directions of changes (except those of $\frac{\partial Q_i}{\partial K}$ and $\frac{\partial p_i}{\partial n}$) are insensitive to either of the two behavioral (Bertrand and Cournot) assumptions.

6. CONCLUDING REMARKS

In this paper we extended the theory of competitive inventory policies to the case of a symmetric oligopoly under a Cournot-like behavioral assumption and a Bertrand-like behavioral assumption. First, in section 2 and 3, we showed how a profit maximizing EOQ model can be formulated for n identical sellers competing for the same potential buyers. From this formulation, symmetric equilibrium conditions were obtained. From these equilibrium conditions and the subsequent sensitivity analysis, following economic relations are derived.

In the Cournot model symmetric equilibrium,

1) given a fixed number of competitors n, as the demand becomes more elastic, the equilibrium price gets closer to the average production cost;
2) given a fixed level of elasticity ϵ, as the number of competitors increases, the price gets closer to the average production cost;
3) the average production cost is equal to the sum of the marginal production cost and the average inventory holding cost;
4) if the inventory holding cost is increased by a small amount, the sales quantity per unit time and the order quantity per cycle will decrease;
5) if the setup cost is increased by a small amount, the sales quantity per unit time will decrease while the order quantity per cycle will increase;

6) if the number of competing seller is increased by a small number, the sales quantity per unit time and the order quantity per cycle will decrease.

It can be easily verified that when n = 1, the EOQ model is analogous to a monopolist's profit maximizing EOQ model insofar as the equilibrium conditions and subsequent sensitivity analysis are concerned. Moreover, when n = 1 and the sales quantity per unit time is constant, the EOQ model is analogous to the traditional EOQ model insofar as the equilibrium conditions and subsequent sensitivity analysis are concerned.

In the Bertrand model symmetric equilibrium,

1) the price is equal to the sum of the per unit production cost and the per unit inventory holding cost while the per unit production cost is equal to the sum of the marginal production cost and the per unit inventory holding cost;
2) if the inventory holding cost is increased by a small amount, the per unit price will increase while the order quantity per cycle will decrease;
3) if the setup cost is increased by a small amount, the per unit price will increase while the order quantity per cycle will increase, remain the same, or decrease, depending upon the conditions given in the Proposition 2;
4) if the number of competing seller is increased by a small number, the per unit price will increase while the order quantity per cycle will decrease;

5) insofar as the directions of changes are concerned, the impacts of Bertrand competition on the equilibrium are analogous to those of the inventory holding cost, but may not be analogous to those of the setup cost;

6) the directions of changes except those of $\frac{\partial Q_i}{\partial K}$ and $\frac{\partial p_i}{\partial n}$ are insensitive to either of the two behavioral (Bertrand and Cournot) assumptions.

It can be easily verified that when $n = 1$, the EOQ model is analogous to a monopolist's profit maximizing EOQ model insofar as the optimality conditions (19) and (20) are concerned. Moreover, when $n = 1$ and the per unit price is constant, the EOQ model is analogous to the traditional EOQ model insofar as the optimality condition (20) is concerned.

The EOQ model developed in this paper is applicable for broad classes of convex cost function $C(\cdot)$ and concave inverse demand function $p(\cdot)$. Our models relate to general practices since numerous industries and firms apply EOQ based decision making under competition. There are several possible extensions that will further improve the relevance of our model to general practices. They include incorporation of more sophisticated features such as quantity discount price schedules, finite production rates, shortages, delivery lags, and promotional (e.g., advertising) effects as well as stochastic demand rates.

From the perspective of game theory, both Cournot Model and Bertrand model in this paper can be considered as only an initial step toward better understanding of competitive inventory policies. It is our hope that more sophisticated equilibrium concepts of game theory (e.g., subgame perfect equilibrium for sequential decisions) will be exploited in the future research on the competitive inventory policies.

ACKNOWLEDGEMENT

Financial supports from Engineering Dean's Office, Engineering Research Institute, and the Office of the Vice Provost for Research of ISU are gratefully acknowledged.

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APPENDIX A: PROOF OF PROPOSITION 1

We note that the sales quantity per unit time and order quantity per cycle (d_i, Q_i) satisfy the equilibrium conditions (10) and (11) and the second order sufficient condition (7) as assumed in Proposition 1. We also note that $C'(\cdot) > 0$, $C''(\cdot) \geq 0$, $p'(\cdot) < 0$, and $p''(\cdot) \leq 0$ while G (defined in section 3) is strictly positive. Moreover, the profit level at (d_i, Q_i) is assumed to be positive. Finally, we note that, for $p(\cdot)$, $p'(\cdot)$, $p''(\cdot)$, $C(\cdot)$, $C'(\cdot)$, and $C''(\cdot)$, the arguments nd_i and Q_i are suppressed. Before we prove Proposition 1, we present the following lemma.

Lemma 1. Under the assumptions of Proposition 1, the following relations hold.

1) $p - (K + C)/Q_i = -p'd_i$

2) $(K + C - Q_iC')/Q_i = hQ_i/(2d_i)$

3) $-p'd_i > hQ_i/(2d_i)$

Proof of Lemma 1. By rearranging the equilibrium conditions (10) and (11), we obtain the relations 1) and 2), i.e.,

$$p - (K + C)/Q_i = -p'd_i \tag{A.1}$$

$$(K + C - Q_iC')/Q_i = hQ_i/(2d_i) \tag{A.2}$$

For relation 3), we note first that the profit relation is given by (from (1) at $d_i + d_{-i} = nd_i$) $pd_i - (K + C)d_i/Q_i - hQ_i/2$. Since the profit is assumed to be positive in the equilibrium for seller i = 1, ⋯, n, we have

$$p \quad - \quad (K + C)/Q_i \quad > \quad hQ_i/(2d_i) \tag{A.3}$$

i.e., per unit revenue - per unit production cost > per unit inventory cost.

By substituting (A.1) into (A.3), we obtain

$$-p'd_i \quad > \quad hQ_i/(2d_i) \tag{A.4}$$

□

Proof of Proposition 1. Let us first consider relations 1) of Proposition 1, $\frac{\partial d_i}{\partial h} < 0$ and $\frac{\partial Q_i}{\partial h} < 0$. From equations (33),

$$\frac{\partial d_i}{\partial h} = G\ (Q_iC' - K - C)/(2Q_i^2) \tag{A.5}$$

From equation (14), we have $Q_iC' - K - C < 0$. Since $G > 0$ and $Q_i > 0$, we have $\frac{\partial d_i}{\partial h} < 0$.

□

Also from equations (33),

$$\frac{\partial Q_i}{\partial h} = G\ (np''d_i + (n+1)p')/2 \tag{A.6}$$

Since $p' < 0$ and $p'' \leq 0$, $np''d_i + (n+1)p' < 0$. Hence, $\frac{\partial Q_i}{\partial h} < 0$.

□

Let us now consider relations 2) of Proposition 1, $\frac{\partial d_i}{\partial K} < 0$ and $\frac{\partial Q_i}{\partial K} > 0$. From equations (35),

$$\frac{\partial d_i}{\partial K} = G\ d_i(-Q_i^2C'' - (K + C - Q_iC'))/Q_i^4 \tag{A.7}$$

Since $C'' \geq 0$ and $K + C - Q_iC' > 0$, $-Q_i^2C'' - (K + C - Q_iC') < 0$.

Hence, $\frac{\partial d_i}{\partial K} < 0$.

□

Also from equations (35),

$$\frac{\partial Q_i}{\partial K} = G\ ((Q_iC' - K - C)/Q_i^3 - (np''d_i + (n+1)p')d_i/Q_i^2) \qquad (A.8)$$

$\frac{\partial Q_i}{\partial K} > 0$ if

$$-(np''d_i + (n+1)p')d_i > (K + C - Q_iC')/Q_i \qquad (A.9)$$

By substituting (A.2) of Lemma 1 into the right hand side of (A.9), we obtain

$$-(np''d_i + (n+1)p')d_i > hQ_i/(2d_i) \qquad (A.10)$$

As for the left hand side of (A.9) (or (A.10)), we have

$$-(np''d_i + (n+1)p')d_i > -p'd_i \qquad (A.11)$$

since $p'' \leq 0$, $p' < 0$, and $n \geq 1$.

Also, by (A.4) of Lemma 1, we have

$$-p'd_i > hQ_i/(2d_i) \qquad (A.12)$$

From (A.10)-(A.12), (A.9) holds. Hence, $\frac{\partial Q_i}{\partial K} > 0$.

□

Finally, let us consider relations 3) of Proposition 1, $\frac{\partial d_i}{\partial n} < 0$ and $\frac{\partial Q_i}{\partial n} < 0$.

From equations (37),

$$\frac{\partial d_i}{\partial n} = -G\ d_i(-Q_i^2C'' - 2(K + C - Q_iC'))((\partial p'/\partial n)d_i+(\partial p/\partial n))/Q_i^3 \qquad (A.13)$$

Since $C'' \geq 0$, $K + C - Q_iC' > 0$, $p'' \leq 0$, and $p' < 0$,

$$(-Q_i^2C'' - 2(K + C - Q_iC'))((\partial p'/\partial n)d_i+(\partial p/\partial n)) > 0.$$

Hence, $\frac{\partial d_i}{\partial n} < 0$.

□

Also from equations (37),

$$\frac{\partial Q_i}{\partial n} = G\ (K + C - Q_iC')((\partial p'/\partial n)d_i+(\partial p/\partial n))/Q_i^2 \qquad (A.14)$$

Since $K + C - Q_iC' > 0$, $p'' \leq 0$, and $p' < 0$, $\frac{\partial Q_i}{\partial n} < 0$ follows.

□

APPENDIX B: PROOF OF PROPOSITION 2

Let us first consider relations 1) of Proposition 1, $\frac{\partial p_i}{\partial h} > 0$ and $\frac{\partial Q_i}{\partial h} < 0$. From equations (42),

$$\frac{\partial p_i}{\partial h} = H\ (d/n)(-Q_i^2 C'' - 2(K + C - Q_i C'))/(2Q_i^2) \qquad (B.1)$$

From equation (28), we have $K + C - Q_i C' > 0$. Since $C'' \geq 0$, $-Q_i^2 C'' - 2(K + C - Q_i C') < 0$ while $H < 0$ by the assumption that $\frac{\partial \pi_i}{\partial p_i} > 0$. Therefore, $\frac{\partial p_i}{\partial h} > 0$.

□

Also from equations (42),

$$\frac{\partial Q_i}{\partial h} = H\ d/(2n) \qquad (B.2)$$

Since $H < 0$, we have $\frac{\partial Q_i}{\partial h} < 0$.

□

Let us now consider relations 2) of Proposition 1. i.e., $\frac{\partial p_i}{\partial K} > 0$ while $\frac{\partial Q_i}{\partial K} > 0$ if $d + d'(p_i - C') > 0$; $\frac{\partial Q_i}{\partial K} = 0$ if $d + d'(p_i - C') = 0$; $\frac{\partial Q_i}{\partial K} < 0$ if $d + d'(p_i - C') < 0$. From equations (44),

$$\frac{\partial p_i}{\partial K} = H\ d^2(-Q_i^2 C'' - 2(K + C - Q_i C'))/(n^2 Q_i^4) \qquad (B.3)$$

Since $-Q_i^2 C'' - 2(K + C - Q_i C') < 0$ and $H < 0$, $\frac{\partial p_i}{\partial K} > 0$.

□

Also from equations (44),

$$\frac{\partial Q_i}{\partial K} = H[\ -d(d + d'(p_i - C'))/(n^2 Q_i^2)\ ] \qquad (B.4)$$

$H(-d) > 0$. Therefore, if $d + d'(p_i - C') > 0$, then $\frac{\partial Q_i}{\partial K} > 0$;

if $d + d'(p_i - C') = 0$, then $\frac{\partial Q_i}{\partial K} = 0$;

if $d + d'(p_i - C') < 0$, then $\frac{\partial Q_i}{\partial K} < 0$.

□

Finally, let us consider relations 3) of Proposition 1, $\frac{\partial p_i}{\partial n} > 0$ and $\frac{\partial Q_i}{\partial n} < 0$.

From equations (46),

$$\frac{\partial p_i}{\partial n} = H\, d^2(-Q_i^2C'' - 2(K + C - Q_iC'))(p_i - (K + C)/Q_i)/(n^3Q_i^3) \quad \text{(B.5)}$$

Since $-Q_i^2C'' - 2(K + C - Q_iC') < 0$ and $p_i - (K + C)/Q_i > 0$, $\frac{\partial p_i}{\partial n} > 0$.

□

Also from equations (46),

$$\frac{\partial Q_i}{\partial n} = H\, d^2(K + C - Q_iC')/(n^3Q_i^2) \quad \text{(B.6)}$$

Since $K + C - Q_iC' > 0$, $\frac{\partial Q_i}{\partial n} < 0$ follows.

□

PAPER 2.

A COMPETITIVE EOQ MODEL
WITH OPTIONS TO REDUCE SETUP AND INVENTORY HOLDING COSTS

A COMPETITIVE EOQ MODEL
WITH OPTIONS TO REDUCE SETUP AND INVENTORY HOLDING COSTS

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ABSTRACT

In this paper, the profit maximizing economic order quantity (EOQ) model is extended to the case of a symmetric oligopoly consisting of several producers who compete with each other for the same potential buyers. For each producer, we assume that the options of investing in reducing the setup and inventory holding costs are available. A primary goal of this paper is to understand economic implications of the resulting equilibrium in terms of critical elements of EOQ models such as the setup and inventory holding costs as well as critical elements of the microeconomic market theory such as the market price and the number of competing producers. For an example, we present a unique insight as to why several Japanese and American producers are striving to reduce the setup costs under ever increasing competition. Specifically, it will be shown that, for a profit maximizing producer, as the number of competing producers increases, his optimal strategy dictates that he reduce his setup and inventory holding costs.

1. INTRODUCTION

This paper extends the profit maximizing economic order quantity (EOQ) model with a constant demand rate over time to the case of a symmetric oligopoly consisting of producers of a single homogeneous product who compete with each other for the same potential buyers. We assume that, for each producer, the options of investing in reducing the setup and inventory holding costs are available. A primary goal of this study is to understand economic implications of the resulting equilibrium in terms of critical elements of EOQ models such as the sales quantity per unit time and the levels of setup and inventory holding costs as well as critical elements of the microeconomic market theory such as the market price and the number of competing producers. For an example, we offer a unique insight as to why several Japanese and American producers are striving to reduce the setup costs under ever increasing competition. Specifically, it will be shown that, for a profit maximizing producer, as the number of competing producers increases (i.e., the competition gets more intense), his optimal strategy dictates that he reduce his setup and inventory holding costs.

The idea of employing profits as a performance measure of EOQ type models has been explored as early as the 1950's (see, e.g., Whitin [29] or Smith [23]). Ladany and Sternlieb [11] not only uses the profit levels as the performance measure, but also provides insights on relations among price, cost, and demand by making the demand dependent on the price and the price dependent on the cost and a fixed mark-up. Brahmbhatt and Jaiswal [3] extends the previous model by incorporating variable mark-up as a function of a capital intensity measure and by maximizing profit over the order quantity and the capital intensity. Arcelus and Srinivasan [1] also extends Ladany and Sternlieb [11] by treating demand as a function of price, price as a function of a variable mark-up rate times a unit cost under profit maximization over the order quantity and the variable mark-up rate. Moreover, Monahan [16] as well as Lal and Staelin [12] developed quantity discount schemes for the seller. Lee and Rosenblatt [13] extended Monahan [16] by incorporating more realistic features (e.g., constraints imposed on the amount of discount that can be offered). The rationale for the quantity discount in these papers is the cost savings resulting from coordination of producers' production quantities and buyers' order quantities under the assumption that both buyers as well as sellers are EOQ based decision makers. The assumption that buyers are EOQ based decision makers is relaxed in a new quantity discount EOQ model in Min [14]. In Min [14], the rationale for the quantity discount is the seller's exploitation of the heterogeneous preferences of the buyers regarding their purchase sizes. More recently, in Min [15], for both uniform pricing and quantity discount pricing (under the heterogeneous buyers' preferences assumption) cases, how to incorporate competition aspects of sellers into EOQ models based on Cournot-like behavioral assumptions (see e.g., Oren, Smith, and Wilson [17] or Varian [28] or Friedman [6]) is discussed.

Also, under the assumption of dynamic and deterministic demands, there have been numerous studies investigating the optimal relations of production schedules, prices, and inventories (see e.g., Gaimon [7], Pekelman [18], Kunreuther and Schrage [10], and Thomas [26]). Thomas [27] investigates the optimal relations of production quantities and prices under the assumption of stochastic demands. Moreover, in Gaimon [8], the assumption of a single firm is replaced by a duopoly, and the optimal relations between production capacities and prices are studied within a differential game framework. Also, in Dockner and Jorgensen [5], optimal pricing strategy under competition is examined and non-cooperative as well as cooperative equilibria results are obtained. In Teng and Thompson [25], an oligopoly model is analyzed and optimal advertising policies are obtained when production costs obey a learning curve. We note that the models constructed and analyzed in the last three papers are also time dependent dynamic models.

Recently, the superiority of an inventory management system called Zero Inventory (often synonymous with Kanban and Just-in-Time; see e.g., Zangwill [30]) has attracted a great deal of attention not only from industries but also from academia. The essential philosophy of Zero Inventory management system is that the inventory results from operational inefficiencies. Hence, the higher the level of inventory, the greater the operational inefficiency. From this perspective, it is well known that several Japanese and American producers strive to reduce the level of inventory as much as possible. In order to reduce the level of inventory, numerous experts in industries and academia find it essential to reduce the setup cost of production. In Porteus [19], such efforts to reduce the setup cost are mathematically incorporated by introducing an investment cost function of reducing the setup cost to undiscounted EOQ models. For the cases of logarithmic investment cost functions and power investment cost functions, his models demonstrate decreased operational costs when the setup cost is reduced. Porteus [20] extends Porteus [19] to the cases of discounted EOQ models. Billington [2] formulates a model of which setup cost is a function of capital expenses and investigates the relations among holding, setup, and capital expenses. In Zangwill [30], however, it is argued via numerical examples that certain efforts to reduce the setup cost will actually increase the operational costs. We note that, in all these papers, the performance criterion has been the minimization of operational costs (as compared to the maximization of profits in our model) and the competition effects on the production and inventory policies are ignored.

In this paper, we construct a model under a Cournot-like behavioral assumption. That is, each producer first predicts his competitors' sales quantities per unit time in maximizing his own profit (see e.g., Oren, Smith, and Wilson [17]). The decision variables of our model are the economic production quantity (in order to be consistent with the term "producer", we will use the term "economic production quantity" in place of "economic order quantity"), the sales quantity per unit time, and the desired levels of setup and inventory holding costs (i.e., the options of investing in reducing setup and inventory holding costs are available to each producer). In our model, we will assume that all critical economic quantities producers must determine such as the optimal economic production quantity and the optimal level of the setup cost are made under the framework of static decision making (cf. dynamic decision making framework; see e.g. [7], [18], [10],and [26]). In order to highlight the optimal relations among the critical economic quantities that are derived under the static decision making framework, we will make the following assumptions. We assume : 1) the demand is deterministic and constant over time; 2) production occurs instantaneously; 3) there is no learning effects in setup or production. Also we will not consider discounting prices and costs over time and other time dependent features such as promotion and advertising. In addition, we will assume that each producer can produce sufficient amount of products to meet any quantity demanded by buyers. Under these assumptions, we formulate a Cournot model consisting of a symmetric oligopoly of n producers with options to invest in reducing setup and inventory holding costs offering a homogeneous product. By a symmetric oligopoly, we mean producers are identical in all economic respects such as production costs and investment costs of reducing setup and inventory holding costs. From the formulation of the Cournot model, we obtain a symmetric equilibrium. The formulation and equilibrium conditions under the Cournot model explicitly depend on n, the number of competing producers in the market. We derive interesting economic implications regarding the market price, demand elasticity, number of competitors, average and marginal costs of production and inventory holding as well as the aforementioned four decision variables. Furthermore, via sensitivity analysis which is based on the equilibrium conditions under the Cournot model, we derive the directions and magnitudes of changes in the aforementioned decision variables with respect to change in the number of competing producers. From the results of the sensitivity analysis, we present several interesting economic implications including a unique insight as to why several Japanese and American producers have devoted so much energy and time to reducing setup costs.

The rest of this paper is organized as follows. In section 2, we formulate the Cournot model and derive and interpret its equilibrium conditions. In section 3, we perform the sensitivity analysis on the Cournot model and interpret its economic implications. Summary and concluding remarks are made in section 4.

2. THE MODEL AND ECONOMIC IMPLICATIONS IN EQUILIBRIUM

We assume that there are n identical producers offering a single homogeneous product. Also we assume that buyers have perfect information about the per unit prices n producers charge. Hence, in equilibrium, all producers will charge the same per unit price, p, the market price. For each producer i, $i = 1, \cdots, n$, as in the cases of traditional EOQ models (see e.g., Hillier and Lieberman [9]), we assume: 1) the goods are produced in equal numbers, Q_i at a time; 2) all Q_i units arrive without delivery lag; 3) no shortage to a buyer is permitted. We also assume that, for each producer i, the total cost per cycle includes the production and inventory holding costs of conventional EOQ models. The production cost per cycle is represented by $K_i + C(Q_i)$ where K_i is the setup cost and $C(Q_i)$ is the production cost incurred in producing Q_i units after the setup. On the other hand, the inventory holding cost is characterized by h, inventory holding cost per unit per unit time. In this paper, the options of investing in reducing the setup cost and the inventory holding cost are available. Specifically, we will characterize these options by defining the following two cost functions (cf. Porteus [19]).

1) $V(K_i;K_0)$: the per unit time cost of reducing the setup cost from the current level of K_0 to the level K_i.

2) $W(h_i;h_0)$: the per unit time cost of reducing the inventory holding cost from the current level of h_0 to the level of h_i.

As implied earlier, we assumed that all cost functions (i.e., $C(Q_i)$, $V(K_i;K_0)$ and $W(h_i;h_0)$) are identical for all producers. We further assume that $C(Q)$ is strictly increasing and convex in Q and $V(K_i; K_0)$ and $W(h_i;h_0)$ are strictly decreasing and concave in K and h, i.e., $C'(Q) > 0$, $C''(Q) \geq 0$, $W'(h_i;h_0) < 0$, $W''(h_i;h_0) \leq 0$, $V'(K_i;K_0) < 0$, and $V''(K_i;K_0) \leq 0$. The convexity of $C(Q)$ implies that the marginal cost of production is increasing in Q where as the concavity of $V(K_i;K_0)$ ($W(h_i;h_0)$) in K_i (h_i) implies that the marginal per unit time cost of reducing the setup cost (or inventory holding cost) with respect to the setup cost (or inventory holding cost) is decreasing.

The sales quantity to buyers per unit time for the entire market is characterized by $d(p)$, a function of per unit market price p. We assume that the sales quantity, given a price, is constant over time. Also we assume that the sales quantity function is strictly decreasing in p, i.e., $d'(p) < 0$. Under the monotonicity assumption of $d'(p) < 0$, the inverse function $p(d)$ exists (with $p'(d) < 0$). The inverse function $p(d)$ specifies the price p that clears d units in the market. We will assume that the inverse function $p(d)$ is concave in d, i.e., $p''(d) \leq 0$. Just as in microeconomic theory (see e.g., Varian [28]), we can refer to $p(d)$ as the inverse demand function while $d(p)$ as the demand function. Since the demand function $d(p)$ is assumed to be constant over time, so is the inverse demand function $p(d)$.

Given the above definitions and assumptions, we develop a Cournot-like framework as follows. We assume that each producer i, $i = 1, \cdots, n$ will predict the total sales quantity per unit time of his n-1 competitors, d_{-i}. Under this prediction, producer i maximizes his profit per unit time over his sales quantity per unit time d_i, economic production quantity per cycle Q_i, desired setup cost per cycle K_i, and desired inventory holding cost per unit per unit time h_i. For the total sales quantity per unit time for the entire market, $d_i + d_{-i}$, the corresponding per unit market price is given by $p(d_i + d_{-i})$. Hence, the total revenue per cycle for producer i is $p(d_i + d_{-i})\, Q_i$. And the corresponding total cost per cycle and the cycle length are given by $K_i + C(Q_i) + h_i Q_i^2/(2d_i) + V(K_i;K_0)Q_i/d_i + W(h_i;h_0)Q_i/d_i$ and Q_i/d_i respectively. Given these expressions for the total revenue, cost, and the cycle length, the problem of maximizing profit per unit time for producer i, π_i, can be stated as follows.

$$\max_{d_i,Q_i,h_i,K_i} \pi_i = p(d_i+d_{-i})d_i - (K_i + C(Q_i))d_i/Q_i - h_iQ_i/2 - W(h_i;h_0) - V(K_i;K_0) \tag{1}$$

The first order optimality conditions of the maximization problem (1) are :

$$\frac{\partial \pi_i}{\partial d_i} = p'(d_i + d_{-i})d_i + p(d_i + d_{-i}) - (K_i + C(Q_i))/Q_i = 0 \tag{2}$$

$$\frac{\partial \pi_i}{\partial Q_i} = -d_i(C'(Q_i)Q_i - K_i - C(Q_i))/Q_i^{\,2} - h_i/2 = 0 \tag{3}$$

$$\frac{\partial \pi_i}{\partial h_i} = -Q_i/2 - W'(h_i;h_0) = 0 \tag{4}$$

$$\frac{\partial \pi_i}{\partial K_i} = -d_i/Q_i - V'(K_i;K_0) = 0 \tag{5}$$

In order to derive the corresponding second order sufficient conditions for optimality, we first obtain the second order derivatives of the profit as follows.

$$\frac{\partial^2 \pi_i}{\partial d_i^2} = p''(d_i + d_{-i})d_i + 2p'(d_i + d_{-i}) \tag{6}$$

$$\frac{\partial^2 \pi_i}{\partial d_i \partial Q_i} = (K_i + C(Q_i) - Q_i C'(Q_i))/Q_i^2 \tag{7}$$

$$\frac{\partial^2 \pi_i}{\partial d_i \partial h_i} = 0 \tag{8}$$

$$\frac{\partial^2 \pi_i}{\partial d_i \partial K_i} = -1/Q_i \tag{9}$$

$$\frac{\partial^2 \pi_i}{\partial h_i \partial Q_i} = -\frac{1}{2} \tag{10}$$

$$\frac{\partial^2 \pi_i}{\partial Q_i^2} = d_i(-Q_i^2 C''(Q_i) - 2(K_i + C(Q_i) - Q_i C'(Q_i)))/Q_i^3 \tag{11}$$

$$\frac{\partial^2 \pi_i}{\partial Q_i \partial K_i} = \frac{d_i}{Q_i^2} \tag{12}$$

$$\frac{\partial^2 \pi_i}{\partial h_i^2} = -W''(h_i;h_0) \tag{13}$$

$$\frac{\partial^2 \pi_i}{\partial h_i \partial K_i} = 0 \tag{14}$$

$$\frac{\partial^2 \pi_i}{\partial K_i^2} = -V''(K_i;K_0) \tag{15}$$

From our assumptions that $p'(\cdot) < 0$ and $p''(\cdot) \leq 0$, we have $\frac{\partial^2 \pi_i}{\partial d_i^2} < 0$. Therefore, the second order sufficient conditions for optimality can be expressed by the following Hessian matrix and the signs of principal minors.

$$\begin{matrix} \nabla_1 < 0 \\ \nabla_2 > 0 \\ \nabla_3 < 0 \\ \nabla_4 > 0 \end{matrix} \begin{bmatrix} \partial^2\pi_i/\partial d_i^2 & \partial^2\pi_i/\partial d_i\partial Q_i & \partial^2\pi_i/\partial d_i\partial h_i & \partial^2\pi_i/\partial d_i\partial K_i \\ \partial^2\pi_i/\partial d_i\partial Q_i & \partial^2\pi_i/\partial Q_i^2 & \partial^2\pi_i/\partial Q_i h_i & \partial^2\pi_i/\partial Q_i\partial K_i \\ \partial^2\pi_i/\partial d_i\partial h_i & \partial^2\pi_i/\partial Q_i\partial h_i & \partial^2\pi_i/\partial h_i^2 & \partial^2\pi_i/\partial h_i\partial K_i \\ \partial^2\pi_i/\partial d_i\partial K_i & \partial^2\pi_i/\partial Q_i\partial K_i & \partial^2\pi_i/\partial h_i\partial K_i & \partial^2\pi_i/\partial K_i^2 \end{bmatrix} \qquad (16)$$

(∇_i defines the ith principal minor)

Throughout the rest of this paper, we will assume that the second order sufficient conditions are satisfied. In addition, we will assume that the resulting profit level of each producer i, i = 1, $\cdots$, n, evaluated at the optimal sales quantity per unit time and production quantity per cycle, desired setup cost per cycle, and desired inventory holding cost per unit per unit time is non-negative (i.e., no producer will exit from the market).

So far we have examined the optimality conditions of a single producer. We now proceed to derive an equilibrium of n producers. Under our assumptions of identical producers, there exists a symmetric equilibrium (see e.g., Oren, Smith, and Wilson [17]) where

$$d_1 = d_2 = \cdots = d_n \tag{17}$$

and

$$Q_1 = Q_2 = \cdots = Q_n \tag{18}$$

i.e., the sales quantity per unit time as well as the economic production quantity are identical for all producers. In this symmetric equilibrium, the total sales quantity per unit time from all competitors of producer i, d_{-i} is equal to $(n-1)d_i$ for $i = 1, \cdots n$. Therefore, the corresponding equilibrium conditions of the optimality conditions (2) - (5) are given by

$$p'(nd_i)d_i + p(nd_i) - (K_i + C(Q_i))/Q_i = 0 \tag{19}$$

$$-d_i(C'(Q_i)Q_i - K_i - C(Q_i))/Q_i^2 - h_i/2 = 0 \tag{20}$$

$$-Q_i/2 - W'(h_i;h_0) = 0 \tag{21}$$

$$-d_i/Q_i - V'(K_i;K_0) = 0 \tag{22}$$

Let us first examine equilibrium condition (19). The corresponding demand elasticity ϵ, $\epsilon = p(d)/(p'(d)d)$ by definition (see e.g., Varian [28]), evaluated at the symmetric equilibrium point becomes:

$$\epsilon = p(nd_i)/(nd_i p'(nd_i)) \tag{23}$$

Hence, in the symmetric equilibrium, equation (19) can be restated as

$$p(nd_i) = \left(\frac{n\epsilon}{n\epsilon + 1}\right)(K_i + C(Q_i))/Q_i \tag{24}$$

Equation (24) states that, given a fixed number of competitors n, as the demand becomes more elastic (i.e., $|\epsilon|$ gets larger), the equilibrium price gets closer to the average production cost. Or as the demand gets more inelastic (i.e., $|\epsilon|$ gets smaller), the equilibrium price gets farther away from the average production cost. If we view the term $\frac{n\epsilon}{n\epsilon + 1}$ as a markup rate, the economic implication is that the markup rate is larger when the demand is more inelastic. On the other hand, given a fixed level of elasticity, ϵ, we observe that as the number of competitors increases (i.e., as the competition gets more intense), the price gets closer to the average production cost. Or as the number of competitors decreases (i.e., as the competition gets less intense), the price gets farther away from the average production cost. We also observe that as the number of competitors decreases, the markup rate increases. In addition, we note that if $-1 < n\epsilon < 0$, the price is negative. Furthermore if $n\epsilon = -1$, it can be easily verified that no production quantity per cycle Q_i satisfies equation (19). Hence, throughout this paper, we limit our analysis to the cases where $n\epsilon < -1$. i.e., $n\epsilon < -1$ will be assumed.

Let us now examine equilibrium condition (20). By rearranging terms of condition (20), we have

$$(K + C(Q_i))/Q_i - C'(Q_i) = hQ_i/(2d_i) \tag{25}$$

Equilibrium condition (25) states that for each producer i, i = 1, $\cdots$, n, the average production cost is equal to the sum of the marginal production cost and the average inventory cost per unit. The economic implication is that the per unit production cost is strictly higher than the per unit inventory cost at the equilibrium since the marginal production cost is assumed to be positive. Also we note that if $(K + C(Q_i))/Q_i \leq C'(Q_i)$, it can be easily verified that no production quantity per cycle Q_i satisfies equation (20). Hence, throughout this paper, we limit our analysis to the cases where $(K + C(Q_i))/Q_i > C'(Q_i)$. i.e., $(K + C(Q_i))/Q_i > C'(Q_i)$ will be assumed.

We now examine equilibrium condition (21). By rearranging terms of condition (21), we have

$$- V'(h_i;h_0) = Q_i/2 \qquad (26)$$

In equation (26), $V'(h_i;h_0)$ denotes the marginal decrease in the per unit time cost of reducing the inventory holding cost (per unit per unit time) with respect to a small increase in h_i where as $Q_i/2$ represents the marginal increase in the per unit time inventory holding cost with respect to a small increase in h_i. Hence, equation (26) states that, in equilibrium, the sum of the marginal decrease in the per unit time cost of reducing the inventory holding cost and the marginal increase in the per unit time inventory holding cost results in zero.

We now proceed to examine equilibrium condition (22). By rearranging terms of condition (22), we have

$$- V'(K_i;K_0) = d_i/Q_i \qquad (27)$$

In equation (27), $V'(K_i;K_0)$ denotes the marginal decrease in the per unit time cost of reducing the setup cost with respect to a small increase in K_i where as d_i/Q_i represents the marginal increase in the per unit time setup cost with respect to a small increase in K_i. Hence, equation (27) states that, in equilibrium, the sum of the marginal decrease in the per unit time cost of reducing the setup cost and the marginal increase in the per unit time inventory holding cost results in zero.

We note that the relations among the equilibrium sales quantity per unit time d_i, the economic production quantity Q_i, the setup cost per cycle K_i and the inventory holding cost per unit per unit time h_i for $i = 1, \cdots, n$ are implicitly determined by (24) to (27). By simultaneously solving conditions (24) to (27) given $p(\cdot)$, $C(\cdot)$, $V(\cdot)$, $W(\cdot)$, h_0, K_0 and n, we can numerically determine the values of d_i, Q_i, K_i, and h_i.

3. SENSITIVITY ANALYSIS

In this section, we investigate the sensitivity of the sales quantity per unit time d_i, production quantity per cycle Q_i, setup cost per cycle K_i, and the inventory holding cost per unit per unit time h_i in equilibrium with respect to the given parameter of our model, the number of competing producers n. Our analysis of sensitivity will be based on differential calculus (especially the implicit function theorem; see e.g., Chiang [4]), which requires variables (or parameters) to be continuous, rather than discrete. Hence, it will be necessary to treat the number of competing producer n ($n \geq 1$), which is hitherto assumed to be an integer, as a continuous variable. The justification for treating n as a continuous variable (to the extent possible) can be found in Seade [21]. The justification in Seade [21] is based on an essential assumption called the "single-signedness" assumption. That is, let us define x(n) to be any relevant function of n (e.g., price p) and let $\nabla x = x(n+1) - x(n)$. Then (sigh ∇x) = (sigh $x'(n)$) is assumed (see e.g., Seade [21] for further details).

At the equilibrium point (d_i, Q_i, h_i, K_i), by applying the implicit function theorem and by allowing n to be continuous, we obtain the following relations for magnitudes of changes $\frac{\partial d_i}{\partial n}$, $\frac{\partial Q_i}{\partial n}$, $\frac{\partial h_i}{\partial n}$, and $\frac{\partial K_i}{\partial n}$ with respect to an infinitesimal increase in the number of competing producers n.

Let F^1, F^2, F^3 and F^4 denote the left hand sides of equilibrium conditions (19) to (22) respectively. From the assumptions that the second order conditions (16) are satisfied, the determinant of the Hessian matrix is positive. It can be easily verified that this implies the determinant of Jacobian of F^1, F^2, F^3 and F^4 with respect to d_i, Q_i, h_i and K_i (shown in the left hand side of equation (28)) is also positive, satisfying a condition necessary for applying the implicit function theorem. Finally, for the inverse demand and cost quantities, $p(\cdot)$, $p'(\cdot)$, $p''(\cdot)$, $C(\cdot)$, $C'(\cdot)$, and $C''(\cdot)$, the arguments nd_i, Q_i, h_i and K_i are suppressed for more comprehensible presentation. Then, we have :

$$\begin{bmatrix} \frac{\partial F^1}{\partial d_i} & \frac{\partial F^1}{\partial Q_i} & \frac{\partial F^1}{\partial h_i} & \frac{\partial F^1}{\partial K_i} \\ \frac{\partial F^2}{\partial d_i} & \frac{\partial F^2}{\partial Q_i} & \frac{\partial F^2}{\partial h_i} & \frac{\partial F^2}{\partial K_i} \\ \frac{\partial F^3}{\partial d_i} & \frac{\partial F^3}{\partial Q_i} & \frac{\partial F^3}{\partial h_i} & \frac{\partial F^3}{\partial K_i} \\ \frac{\partial F^4}{\partial d_i} & \frac{\partial F^4}{\partial Q_i} & \frac{\partial F^4}{\partial h_i} & \frac{\partial F^4}{\partial K_i} \end{bmatrix} \begin{bmatrix} \frac{\partial d_i}{\partial n} \\ \frac{\partial Q_i}{\partial n} \\ \frac{\partial h_i}{\partial n} \\ \frac{\partial K_i}{\partial n} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial n} \\ -\frac{\partial F^2}{\partial n} \\ -\frac{\partial F^3}{\partial n} \\ -\frac{\partial F^4}{\partial n} \end{bmatrix} \tag{28}$$

where the elements of equation (28) are as follows.

$$\frac{\partial F^1}{\partial d_i} = nd_i p'' + (n+1)p' \tag{29}$$

$$\frac{\partial F^1}{\partial Q_i} = \frac{\partial F^2}{\partial d_i} = \frac{K_i + C - Q_i C'}{Q_i^2} \tag{30}$$

$$\frac{\partial F^1}{\partial h_i} = \frac{\partial F^3}{\partial d_i} = 0 \tag{31}$$

$$\frac{\partial F^1}{\partial K_i} = \frac{\partial F^4}{\partial d_i} = -\frac{1}{Q_i} \tag{32}$$

$$\frac{\partial F^2}{\partial h_i} = \frac{\partial F^3}{\partial Q_i} = -\frac{1}{2} \tag{33}$$

$$\frac{\partial F^2}{\partial Q_i} = \frac{d_i(-Q_i^2C'' - 2(K_i + C - Q_iC'))}{Q_i^3} \tag{34}$$

$$\frac{\partial F^2}{\partial K_i} = \frac{\partial F^4}{\partial Q_i} = \frac{d_i}{Q_i^2} \tag{35}$$

$$\frac{\partial F^3}{\partial h_i} = -W''(h_i;h_0) \tag{36}$$

$$\frac{\partial F^3}{\partial K_i} = \frac{\partial F^4}{\partial h_i} = 0 \tag{37}$$

$$\frac{\partial F^4}{\partial K_i} = -V''(K_i;K_0) \tag{38}$$

$$\frac{\partial F^2}{\partial n} = \frac{\partial F^3}{\partial n} = \frac{\partial F^4}{\partial n} = 0 \tag{39}$$

$$\frac{\partial F^1}{\partial n} = d_i\frac{\partial p'}{\partial n} + \frac{\partial p}{\partial n} \tag{40}$$

The inverse of the Jacobian matrix exists since its determinant is nonzero. Hence, we solve the equation (28) for the magnitudes of changes $\frac{\partial d_i}{\partial n}$, $\frac{\partial Q_i}{\partial n}$, $\frac{\partial K_i}{\partial n}$, and $\frac{\partial h_i}{\partial n}$ as follows.

In the following derivations, the quantity I is defined to be the inverse of the determinant of the Jacobian matrix in the left hand side of equation (28). After some matrix operations, we have :

$$\frac{\partial d_i}{\partial n} = -I\left(d_i\frac{\partial p'}{\partial n} + \frac{\partial p}{\partial n}\right)\left[W''(h_i;h_0)V''(K_i;K_0)d_i(-Q_i^2C''-2(K_i+C-Q_iC'))/Q_i^3 + d_i^2W''(h_i;h_0)/Q_i^4 + V''(K_i;K_0)/4\right] \tag{41}$$

$$\frac{\partial Q_i}{\partial n} = I(d_i\frac{\partial p'}{\partial n} + \frac{\partial p}{\partial n})\Big[W''(h_i;h_0)V''(K_i;K_0)(K_i+C-Q_iC')/Q_i^2 - W''(h_i;h_0)d_i/Q_i^3\Big] \quad (42)$$

$$\frac{\partial h_i}{\partial n} = -I(d_i\frac{\partial p'}{\partial n} + \frac{\partial p}{\partial n})\Big[V''(K_i;K_0)(K_i+C-Q_iC')/(2Q_i^2) - d_i/(2Q_i^3)\Big] \quad (43)$$

$$\frac{\partial K_i}{\partial n} = I(d_i\frac{\partial p}{\partial n} + \frac{\partial p}{\partial n})\Big[W''(h_i;h_0)d_i(K_i+C-Q_iC')/Q_i^4 + W''(h_i;h_0)d_i(-Q_i^2C''-2(K_i+C-Q_iC'))/Q_i^4 + 1/(4Q_i)\Big] \quad (44)$$

The corresponding directions of $\frac{\partial d_i}{\partial n}$, $\frac{\partial Q_i}{\partial n}$, $\frac{\partial h_i}{\partial n}$, and $\frac{\partial K_i}{\partial n}$ are summarized in the following proposition (see Appendix for the proof).

Proposition 1 : *Assume that the sales quantity per unit time ,production quantity per cycle and the levels of setup and inventory holding costs* (d_i, Q_i, h_i, K_i) *satisfy the equilibrium conditions* (19) *to* (22) *and the second order sufficient condition* (16). *Assume further that for the production cost, the investment cost and the inverse demand functions,* $C'(Q) > 0$, $C''(Q) \geq 0$, $V'(K) < 0$, $W'(h) < 0$, $V''(K) \leq 0$, $W''(h) \leq 0$, $p'(d) < 0$, *and* $p''(d) \leq 0$. *Moreover, assume that the profit level at* (d_i, Q_i, h_i, K_i) *is non-negative.*
Then, we have :

$$\frac{\partial d_i}{\partial n} < 0,\ \frac{\partial Q_i}{\partial n} < 0,\ \frac{\partial h_i}{\partial n} < 0 \text{ and } \frac{\partial K_i}{\partial n} < 0.$$

The economic implications of Proposition 1 are as follows. In the equilibrium, under the aforementioned single-signedness assumption, we conclude that if the number of competing producers increases by a smaller number, the sales quantity per unit time, the production quantity per cycle, the inventory holding cost per unit per unit time and the setup cost per cycle will decrease in equilibrium. We note that the change in the frequency of production is indeterminate (i.e., the corresponding cycle may be longer or shorter). The sign of the corresponding change in the per unit price (i.e., $p(nd_i)$), however, is also indeterminate due to the following reason. Let $\hat{d}_i$ ($\hat{d}_i < d_i$) denote the new sales quantity per unit time for producer i and let $\hat{n}$ ($\hat{n} > n$) denote the new number of competing producers. Then, the corresponding new per unit price is given by $\hat{p} = p(\hat{n}\hat{d}_i)$. Since $\hat{n} > n$ and $\hat{d}_i < d_i$, the sign of $p(\hat{n}\hat{d}_i) - p(nd_i)$ is indeterminate. i.e., the per unit price may increase, remain the same, or decrease when the number of competing producers increased by a small number.

From the perspective of investing in setup and inventory holding costs, the fact that $\frac{\partial K_i}{\partial n} < 0$ and $\frac{\partial h_i}{\partial n} < 0$ in the equilibrium implies the following. For a profit maximizing producer, as the number of competing producers increases (i.e., the competition gets more intense), his optimal strategy dictates that he reduce his setup and inventory holding costs.

4. CONCLUDING REMARKS

In this paper, we extended the profit maximizing EOQ model by introducing competition aspects under a Cournot-like behavioral assumption and by treating the setup cost and inventory holding cost as decision variables. First we showed how a profit maximizing EOQ model can be formulated for n identical producers competing for the same potential buyers. From this formulation, we obtained symmetric equilibrium conditions. From these equilibrium conditions and the subsequent sensitivity analysis, interesting economic relations are obtained.

From the perspective of Zero Inventory Philosophy, this paper provided an additional insight as to why several Japanese and American producers strive to reduce the setup cost. That is, as the number of competing producers increases (i.e., the competition gets more intense), the optimal strategy of a profit maximizing producer dictates that he invest in reducing setup and inventory holding costs.

The EOQ model developed in this paper is applicable for broad classes of convex $C(\cdot)$ function, concave $V(\cdot)$ and $W(\cdot)$ functions, and concave $p(\cdot)$ function. Our model relates general practices since numerous industries and firms apply EOQ based decision making under competition. There are several possible extensions that will further improve the relevance of our model to general practices. They include incorporation of more sophisticated features such as quantity discount price schedules, finite production rates, shortages, delivery lags, and promotional (e.g., advertising) effects as well as stochastic demand rates. From the perspective of Zero Inventory Philosophy, it would be of interest to study the effects of competition on process quality improvement and effective capacity in conjunction with the setup cost reduction (see e.g., Porteus [21] and Spence and Porteus [24]).

From the perspective of game theory, Cournot Model in this paper can be considered as only an initial step toward better understanding of competitive inventory policies. It is our hope that more sophisticated equilibrium concepts of game theory (e.g., subgame perfect equilibrium for sequential decisions) will be exploited in the future research on the competitive inventory policies.

ACKNOWLEDGEMENTS

This research was supported by a grant from the Engineering Research Institute at the Iowa State University.

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APPENDIX: PROOF OF PROPOSITION 1

From equation (41), we have

$$\frac{\partial d_i}{\partial n} = -I(d_i\frac{\partial p'}{\partial n}+\frac{\partial p}{\partial n})\Big[W''(h_i;h_0)V''(K_i;K_0)d_i(-Q_i^2C''-2(K_i+C-Q_iC'))/Q_i^3 + d_i^2W''(h_i;h_0)/Q_i^4 + V''(K_i;K_0)/4\Big] \quad (A.1)$$

From equation (25), we have $K_i + C - Q_iC' > 0$. Since $C'' \geq 0$, $-Q_i^2C'' - 2(K_i + C - Q_iC') < 0$, $W''(h_i) < 0$ and $V''(K_i) < 0$, we have $\Big[\ \cdot\ \Big]$ is less than zero. Since $(d_i\frac{\partial p'}{\partial n} + \frac{\partial p}{\partial n}) < 0$ and $I > 0$, we obtain $\frac{\partial d_i}{\partial n} < 0$. □

From (42), we have

$$\frac{\partial Q_i}{\partial n} = I(d_i\frac{\partial p'}{\partial n}+\frac{\partial p}{\partial n})\Big[W''(h_i;h_0)V''(K_iK_0)(K_i+C-Q_iC'))/Q_i^2 - W''(h_i;h_0)d_i/Q_i^3\Big] \quad (A.2)$$

From equation (25), we have $K_i + C - Q_iC' > 0$. Since $W''(h_i) < 0$ and $V''(K_i) < 0$, we have $\Big[\ \cdot\ \Big]$ is less than zero.
Since $(d_i\frac{\partial p'}{\partial n} + \frac{\partial p}{\partial n}) < 0$ and $I > 0$, we obtain $\frac{\partial Q_i}{\partial n} < 0$. □

From (43), we have

$$\frac{\partial h_i}{\partial n} = -I(d_i\frac{\partial p'}{\partial n} + \frac{\partial p}{\partial n})\Big[V''(K_i;K_0)(K_i+C-Q_iC')/(2Q_i^2) - d_i/(2Q_i^3)\Big] \quad (A.3)$$

From equation (25), we have $K_i + C - Q_iC' > 0$. Since $V''(K_i) < 0$, we have $\Big[\ \cdot\ \Big]$ is less than zero.
Since $(d_i\frac{\partial p'}{\partial n} + \frac{\partial p}{\partial n}) < 0$ and $I > 0$, we obtain $\frac{\partial h_i}{\partial n} < 0$. □

From (44), we have

$$\frac{\partial K_i}{\partial n} = I(d_i\frac{\partial p}{\partial n} + \frac{\partial p}{\partial n})\left[W''(h_i;h_0)d_i(-Q_i^2C''-(K_i+C-Q_iC'))/Q_i^4 + 1/(4Q_i)\right] \tag{A.4}$$

From equation (25), we have $K_i + C - Q_iC' > 0$. Since $C'' \geq 0$, $-Q_i^2C'' - 2(K_i + C - Q_iC') < 0$, $W''(h_i) < 0$, we have $\left[\cdot \right]$ is greater than zero.

Since $(d_i\frac{\partial p'}{\partial n} + \frac{\partial p}{\partial n}) < 0$ and $I > 0$, we obtain $\frac{\partial K_i}{\partial n} < 0$. □

PAPER 3.

OPTIMAL SELLING QUANTITY AND PURCHASING PRICE FOR INTERMEDIARY FIRMS

OPTIMAL SELLING QUANTITY AND PURCHASING PRICE FOR INTERMEDIARY FIRMS

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ABSTRACT

Intermediary firms are economic agents that purchase from mostly small and numerous independent producers and sell to other firms or to the public. In this paper, how intermediary firms can optimally determine both selling quantity and purchasing price of a product is investigated. By incorporating the special structure of intermediary firms' environments and by modifying the conventional economic order quantity (EOQ) model accordingly, we provide optimal decision rules regarding the selling quantity and purchasing price for intermediary firms.
(Economic Order Quantity, Pricing)

INTRODUCTION

The conventional economic order quantity (EOQ) and economic production quantity (EPQ) have been extensively studied and continually modified in order to accommodate specific business needs and environments[1,2]. In this paper, we extend the conventional EOQ/EPQ models so as to determine the optimal selling quantity and purchasing price for an intermediary firm. We define an intermediary firm to be an economic agent that purchases products from numerous independent producers and sells those purchased products to other firms that process or utilize the products (or to the public) at a given market price. Such firms can be found in numerous industries. For examples, there are 1) garment and apparel industry firms that purchase piece works ("homework") from independent sewers, and 2) agricultural industry firms that purchase dairy and other agricultural products from independent farmers.

The objective function employed in this paper is that of profit maximization. The idea of employing profits as a performance measure of EOQ type models has been explored as early as in the 1950's[3]. Ladany and Sternlieb[4] not only uses the profit levels as the performance measure, but also provides insights on relations among price, cost, and demand by making the demand dependent on the price and the price dependent on the cost. Arcelus and Srinivasan[5] extend Ladany and Sternlieb's work by exploring alternative investment oriented performance measures such as return on investment and residual income. Also, we employ the inspection cost feature of conventional EOQ/EPQ models[6] in order to account for possible defective products from independent producers.

The rest of this paper is organized as follows. We first define the special structure of intermediary firms' environments and formulate the basic profit maximization model over the selling quantity, given a fixed purchasing price. Next we extend the basic model by making the fixed purchasing price as a variable. Also, we add to the basic model an inspection cost component, which is a realistic feature for an intermediary firm. Finally, an illustrative example is provided and concluding marks are made.

DESCRIPTION OF INTERMEDIARY FIRMS' ENVIRONMENTS

Let us denote the unit price an intermediary firm pays to independent producers by r for a single type of product (e.g., eggs or milk). The annual supply rate of the product to the intermediary firm from the independent producers at price r is denoted by s(r). In the traditional economic production quantity (EPQ) perspective, s(r) can be viewed as the production rate. In this paper, we will assume that the annual rate is constant over time and there is a linear relation between r and s(r). That is, s(r) = gr, where g is a positive proportionality constant. Given a supply proportionality constant g, a higher price r implies a higher supply rate s(r). Also given a fixed price r, a higher supply proportionality constant g implies a higher supply rate s(r). In the basic model, we will assume that the price r is fixed and relax this assumption later. The purchased units are stored in the firm at a cost of rF per unit per annum where F is the annual holding cost as a fraction of unit purchasing cost to the intermediary firms (see e.g., Hax and Candea[7] for the various components of the inventory holding cost and the role of holding cost F). Once an amount of Q units accumulates, all Q units are sold to another firm that processes or utilizes the products (or to the public) at a given market price of p per unit. The cost incurred to the intermediary firm in selling the accumulated products is represented by a fixed selling cost K (for arranging transportation, etc.) and a variable selling cost c per unit (for actual transportation, etc.).

BASIC MODEL

Under our definitions and assumptions, the revenue per cycle is given by pQ while the payment to the independent producers per cycle is given by rQ. The total selling cost per cycle is K + cQ and the inventory holding cost per cycle is $rFQ^2/(2s(r))$. Therefore, the profit per cycle, PRC, which is the revenue less the cost, is given by

$$PRC = pQ - rQ - K - cQ - rFQ^2/(2s(r)) \qquad (1)$$

The corresponding annual profit, PRA, can be obtained from equation (1) by dividing PRC by Q/s(r), the cycle length (period). Namely,

$$PRA = ps(r) - rs(r) - Ks(r)/Q - cs(r) - rFQ/2 \qquad (2)$$

In order to obtain the profit maximizing selling quantity Q^*, PRA is differentiated with respect to Q and set equal to zero. Hence,

$$Kgr/Q^2 - rF/2 = 0 \qquad (3)$$

given s(r) = gr.

From equation (3), the optimal selling quantity Q^* is given by

$$Q^* = (2Kg/F)^{0.5} \qquad (4)$$

From equation (4), the optimal selling quantity Q^* decreases as the annual holding cost F increases. On the other hand, the optimal selling quantity Q^* increases as the fixed selling cost K or the supply proportionality constant g increases.

OPTIMAL SELLING QUANTITY AND PURCHASING PRICE UNDER BASIC MODEL

Let us now relax the assumption that the purchasing price r is fixed. Instead, in this subsection, we will assume that the intermediary firm can choose the unit price to the independent producers (i.e., r is a variable). Hence, we are maximizing PRA with respect to both r and Q simultaneously. By differentiating (2) with respect to Q and r and by setting the differentiated quantities equal to zero, we obtain two equations relating Q and r with other parameters (P, K, F, g, and c). Solving these two equations for the optimal selling quantity Q^* and purchasing price r^*. we find that Q^* is identical to that of the basic problem shown in equation (4). Moreover, the corresponding optimal purchasing price r^* is given by

$$r^* = p/2 - (KF/(2g))^{0.5} - c/2 \tag{5}$$

From equation (5), we obtain intuitive results that the optimal purchasing price r^* increases as the selling price p or supply proportionality constant g increases. On the other hand, the optimal purchasing price r^* decreases as the fixed selling cost K, variable selling cost c, or annual holding cost F increases.

BASIC MODEL WITH INSPECTION COST

In this subsection, we make the following assumption : Prior to purchase by the intermediary firm, each unit of products is inspected for possible defectiveness at a cost of i per unit. We will assume that a fraction b (defect-rate) of s(r) is defective and the intermediary firm pays only for non-defective units. We will also assume that the intermediary firm determines both the selling quantity Q and the purchasing price r simultaneously. Under this additional assumption, the revenue per cycle is given by pQ while the payment to the independent producers per cycle is given by rQ. Also the total amount of supply including defective items per cycle is given by Q/(1-b). Hence, the total inspection cost per cycle is iQ/(1-b). The total selling cost per cycle is K + cQ and the inventory holding cost per cycle is $rFQ^2/(2(1-b)s(r))$. Therefore, the profit per cycle, PRC, which is the revenue less the cost, is given by

$$PRC = pQ - rQ - iQ/(1-b) - K - cQ - rFQ^2/(2(1-b)s(r)) \qquad (6)$$

The cycle length (period) is given by Q/((1-b)s(r)). Dividing PRC by this cycle length, we obtain the corresponding annual profit, PRA, as below.

$$PRA = p(1-b)s(r) - r(1-b)s(r) - is(r) - (1-b)Ks(r)/Q - c(1-b)s(r) - rFQ/2 \qquad (7)$$

In order to obtain the profit maximizing selling quantity Q^* and purchasing price r^* from (7), we perform a sequence of operations analogous to the one shown in the previous subsection. The resulting Q^* and r^* are given by

$$Q^* = (2(1-b)Kg/F)^{0.5} \qquad (8)$$

$$r^* = p/2 - (KF/(2(1-b)g)^{0.5} - i/(2(1-b)) - c/2 \qquad (9)$$

From equation (8), the optimal selling quantity Q^* decreases as the annual holding cost F or defect-rate b increases. On the other hand, the optimal selling quantity Q^* increases as the fixed selling cost K or the supply proportionality constant g increases. From equation (9), we obtain intuitive results that the optimal purchasing price r^* increases as the selling price p or supply proportionality constant g increases. On the other hand, the optimal purchasing price r^* decreases as the fixed selling cost K, variable selling cost c, per unit inspection cost i, defect-rate b, or annual holding cost F increases.

AN ILLUSTRATIVE EXAMPLE

We solve a profit maximization problem over the selling quantity Q and purchasing price r to illustrate some of the features discussed. Let us assume the following values are provided either by estimations from free market or by regulatory rules.

$K = 1$

$p = 10$

$F = 0.05$

$g = 0.5$

$c = 0.5$

$i = 0.1$

$b = 0.05$

From equation (8) and (9), the optimal selling quantity and purchasing price are given by

$Q^* = 4.359$

$r^* = 4.468$

It can be easily verified that the corresponding annual profit and the optimal cycle length are 9.482 and 2.054 respectively.

CONCLUDING REMARKS

We have shown how to formulate the profit maximization problem for intermediary firms utilizing the special structure of the firms' environments. The optimal selling quantity and purchasing price are derived in terms of fixed and variable selling costs, supply proportionality constant, annual holding cost, selling price, inspection cost, and defect-rate.

The observation that the supply rate s(r) depends on the purchasing price r is a prevalent feature in numerous other kinds of firms. For example, in order to operate efficiently, various types of processing and manufacturing firms (i.e., firms that process supplied inputs into different outputs as opposed to intermediary firms that accumulate supplied inputs and sell them to other firms) must take this relation between the supply rate of inputs and their corresponding prices into account. For such firms, the model in this paper can be a basis for further research.

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PAPER 4.

AN ANALYSIS OF OPTIMAL INVENTORY AND PRICING POLICIES UNDER LINEAR DEMAND

AN ANALYSIS OF OPTIMAL INVENTORY AND PRICING POLICIES UNDER LINEAR DEMAND

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ABSTRACT

In this paper, for a single seller, we compare and contrast the optimal inventory and pricing policies under profit maximization vs. ROII (return on inventory investment) maximization when demand is linear in price. By studying the optimality conditions and the corresponding closed-form optimal solutions, several interesting economic implications are derived. In particular, we show that when a cost factor such as the setup cost, inventory holding cost per unit per unit time, or per unit ordering cost after the setup is sufficiently high, the choice of the objective between profit maximization and ROII maximization is inconsequential to the seller in so far as his optimal decisions are concerned.

INTRODUCTION

In a recent paper by Rosenberg [5], for a single seller, optimal price-inventory decisions in the face of alternative criteria are studied for logarithmic concave demand functions (which include linear demand functions). The alternative models studied are a profit maximizing EOQ type model (the profit maximization model), an ROII (return on inventory investment) maximizing EOQ type model (the ROII maximization model), and an Economic Theory of the Firm model (the ETF model; a profit maximizing model without setup and inventory holding costs). In particular, under the linear demand assumption, these three models are analyzed in detail and numerical examples are presented. In the analysis, closed-form optimal solutions are employed for the ROII maximization model and the ETF model while an examination of optimality conditions and an iterative procedure (e.g., the Newton-Raphson method) are employed for the profit maximization model.

In this paper, however, for the profit maximization model, the closed-form optimal solution is employed for the analysis, which is attainable directly from the optimality conditions. The closed-form optimal solution for the profit maximization model enables us to perform more comprehensive and tangible analysis than the analysis shown in [5] under the assumption of linear demand in so far as the profit maximization model and ROII maximization model are concerned.

Specifically, in this paper, we compare and contrast the optimal inventory and pricing policies for a single seller under profit maximization vs. ROII maximization when demand is linear. First, we formulate the profit maximization model and the ROII maximization model and derive the corresponding closed-form optimal solutions from the optimality conditions of the two models. Next, we obtain the relative bounds of the optimal decisions of the two models by examining the magnitudes of the closed-form optimal solutions. In addition, by studying the optimality conditions of the two models, we derive interesting relations among the price, average ordering cost, price elasticity of demand, and markup rate. Finally, we investigate the sensitivity of the optimal decisions with respect to the choice of the objective. In particular, we show that when a cost factor (e.g., the setup cost) is sufficiently high, the choice of the objective between profit maximization and ROII maximization is inconsequential to the seller in so far as his optimal decisions are concerned.

Throughout this paper, we assume that the seller will not operate (i.e., the seller will exit from the market) if his optimal profit (ROII) level is strictly negative under profit (ROII) maximization. Hence, we will consider only the cases where the optimal profit (ROII) level under profit (ROII) maximization is non-negative. i.e., a non-negative optimal profit (ROII) level under profit (ROII) maximization is assumed for the analysis.

BASIC MODELS

We define the following variables and parameters for our models.

Q : the order quantity.

d : the demand per unit time.

p : the per unit price that clears d units in the market;

$p(d) = \alpha - \beta d,\ d \in [0,\ \alpha/\beta]$.

K : the set up cost.

c : the per unit ordering cost after the setup.

h : the inventory holding cost per unit per unit time.

h′: the inventory holding cost per unit per unit time excluding any opportunity cost; $h' < h$ (i.e., a positive opportunity cost is assumed).

T : the cycle length.

ϵ : the price elasticity of demand; $\epsilon = \frac{dd(p)}{dp}\frac{p}{d}$.

π : the profit per unit time.

R : the return on inventory investment (ROII).

In addition, throughout this paper, as in the conventional EOQ models (see e.g., Hillier and Lieberman [2]), we will assume that 1) the demand is constant over time given a price p; 2) the goods are ordered in equal quantities, Q at a time; 3) all Q units arrive without delivery lag; 4) no shortage is allowed. We now derive the optimal solutions for the profit maximization model under linear demand.

The Profit Maximization Model

Under our definitions and assumptions, the total revenue per cycle, the total cost per cycle, and the cycle length are given by $p(d)Q$, $K + cQ + hQ^2/(2d)$, and Q/d, respectively. Hence, the corresponding profit per cycle and the profit per unit time are given by $p(d)Q - K - cQ - hQ^2/(2d)$ and $p(d)d - Kd/Q - cd - hQ/2$, respectively. Since $p(d) = \alpha - \beta d$, the profit per unit time (denoted by π) maximization problem is formulated as follows.

$$\max_{d,Q} \pi = d(\alpha - \beta d - c) - Kd/Q - hQ/2 \qquad (1)$$

The corresponding first order necessary conditions are given by

$$\frac{\partial \pi}{\partial d} = \alpha - c - 2\beta d - K/Q = 0 \qquad (2)$$

$$\frac{\partial \pi}{\partial Q} = Kd/Q^2 - h/2 = 0 \qquad (3)$$

By substituting and rearranging the relation $Q = (2Kd/h)^{0.5}$ from (3) into (2), we obtain the optimality condition for d as follows:

$$d^{1.5} + \frac{(c-\alpha)}{2\beta} d^{0.5} + \left(\frac{hK}{8\beta^2}\right)^{0.5} = 0 \qquad (4)$$

By employing the trigonometric methods (see e.g., Chapter 3 of Mishina and Proskuyakov [3], Chapter 2 of Griffiths [1], or appendix of Porteus [4]), we obtain the optimal demand per unit time, d_π, as follows.

$$d_\pi = \frac{2(\alpha-c)}{3\beta}\cos^2\left(\frac{\theta}{3}\right) \tag{5}$$

where $\cos\theta = -\left(\frac{27\beta hK}{4(\alpha-c)^3}\right)^{0.5}$, and $\pi/2 < \theta \leq 3\pi/4$.

We note that the upper bound of $3\pi/4$ on the critical angle θ is obtained from the assumption that the resulting profit per unit time is non-negative. On the other hand, the lower bound of $\pi/2$ on the critical angle θ implies that parameters β, h, and K should all be strictly positive in order for the profit maximization EOQ model to be non-degenerate. From (5), the corresponding order quantity Q_π is:

$$Q_\pi = \left(\frac{4K(\alpha-c)}{3\beta h}\right)^{0.5}\cos\left(\frac{\theta}{3}\right) \tag{6}$$

for $\pi/2 < \theta \leq 3\pi/4$ and $\cos\theta = -\left(\frac{27\beta hK}{4(\alpha-c)^3}\right)^{0.5}$

For $\pi/2 < \theta \leq 3\pi/4$, it can be easily verified that the second order sufficient conditions for the profit maximization are satisfied at (d_π, Q_π) given by expressions (5) and (6).

From (5) and (6), we obtain the corresponding optimal price p_π and cycle length T_π as follows.

For $\pi/2 < \theta \leq 3\pi/4$ and $\cos\theta = -\left(\frac{27\beta hK}{4(\alpha-c)^3}\right)^{0.5}$,

$$p_\pi = \alpha - \frac{2(\alpha-c)}{3}\cos^2\left(\frac{\theta}{3}\right) \tag{7}$$

$$T_\pi = \left(\frac{3K\beta}{h(\alpha-c)}\right)^{0.5}\left(\cos\left(\frac{\theta}{3}\right)\right)^{-1} \tag{8}$$

Given the optimal quantities (5)-(8), the corresponding profit and ROII levels evaluated at the optimal quantities (5)-(8), π_π and R_π (see the next subsection for the derivation of ROII), are obtained as below.

For $\pi/2 < \theta \leq 3\pi/4$ and $\cos\theta = -\left(\frac{27\beta hK}{4(\alpha-c)^3}\right)^{0.5}$,

$$\pi_\pi = \frac{(\alpha-c)^2}{\beta}\left(\tfrac{4}{3}\cos^4(\tfrac{\theta}{3}) - \tfrac{2}{3}\cos^2(\tfrac{\theta}{3})\right) \tag{9}$$

$$R_\pi = \frac{2h}{c}\left(\cos^2(\tfrac{\theta}{3})\right)/(3 - 4\cos^2(\tfrac{\theta}{3})) - \frac{h'}{c} \tag{10}$$

The ROII Maximization Model

Analogous to the case of the profit maximization model, the total revenue per cycle, the total cost per cycle excluding any opportunity cost in the inventory holding cost h (i.e., h is now replaced by h′), and the cycle length are given by $p(d)Q$, $K + cQ + h'Q^2/(2d)$, and Q/d, respectively. Hence, the corresponding profit per cycle excluding any opportunity cost in the inventory holding cost h and the profit per unit time excluding any opportunity cost in the inventory holding cost h are given by $p(d)Q - K - cQ - h'Q^2/(2d)$ and $p(d)d - Kd/Q - cd - h'Q/2$, respectively. The value of inventory investment per unit time is given by $cQ/2$ because the amount of inventory per unit time is $Q/2$ and the per unit cost of ordering after the setup is c. The return on inventory investment (ROII) is defined to be the ratio of profit per unit time excluding any opportunity cost in the inventory holding cost h to the value of inventory investment per unit time, $cQ/2$. Hence, the ROII for the seller can be obtained by dividing $p(d)d - Kd/Q - cd - h'Q/2$ by $cQ/2$. Given the relation $p(d) = \alpha - \beta d$, the resulting ROII maximization problem can be stated as follows.

$$\max_{d,Q} R = 2d(\alpha - \beta d - c)/(cQ) - 2Kd/(cQ^2) - h'/c \tag{11}$$

The corresponding first order necessary conditions are given by

$$\frac{\partial R}{\partial d} = 2(\alpha - 2\beta d - c)/(cQ) - 2K/(cQ^2) = 0 \tag{12}$$

$$\frac{\partial R}{\partial Q} = - 2d(\alpha - \beta d - c)/(cQ^2) + 4Kd/(cQ^3) = 0 \tag{13}$$

By solving equations (12) and (13) for the optimal demand per unit time d_R and the optimal order quantity Q_R, we obtain the following expressions.

$$d_R = (\alpha - c)/(3\beta) \tag{14}$$

$$Q_R = 3K/(\alpha - c) \tag{15}$$

where $\frac{2(\alpha - c)^3}{27\beta K} \geq h'$.

The condition $\frac{2(\alpha - c)^3}{27\beta K} \geq h'$ implies that the return on inventory investment is non-negative. Also, it can be easily verified that the second order sufficient conditions for the ROII maximization are satisfied at (d_R, Q_R) given by expressions (14) and (15). Given optimal quantities (14) and (15), we can obtain the corresponding optimal price p_R and cycle length T_R as below.

$$p_R = (c + 2\alpha)/3 \tag{16}$$

$$T_R = 9\beta K/(\alpha - c)^2 \tag{17}$$

Given the optimal quantities (14)-(17), the corresponding profit and ROII levels evaluated at the optimal quantities (14)-(17), π_R and R_R, are obtained as below.

$$\pi_R = (\alpha-c)^2/(9\beta) - 3hK/(2(\alpha-c)) \tag{18}$$

$$R_R = 2(\alpha-c)^3/(27\beta cK) - h'/c \tag{19}$$

COMPARATIVE ANALYSIS OF OPTIMAL POLICIES: PROFIT VS. ROII

Relative Bounds of the Optimal Solutions

In this subsection, let us first compare the relative magnitudes of the optimal order quantities Q_π and Q_R. From equations (6) and (15), the ratio of $\frac{Q_R}{Q_\pi}$ is given by

$$\frac{Q_R}{Q_\pi} = (3K/(a - c))/ \left(\left(\frac{4K(a-c)}{3\beta h}\right)^{0.5}\cos\left(\frac{\theta}{3}\right)\right) \tag{20}$$

$$\text{where } \cos\theta = -\left(\frac{27\beta hK}{4(a-c)^3}\right)^{0.5} \text{ for } \pi/2 < \theta \leq 3\pi/4.$$

The right hand side of equation (20) can be rearranged such that

$$\frac{Q_R}{Q_\pi} = \left(\frac{27\beta hK}{4(a-c)^3}\right)^{0.5}/\cos\left(\frac{\theta}{3}\right)$$
$$= -\cos\theta/\cos\left(\frac{\theta}{3}\right) \tag{21}$$

By employing the identity relation, $\cos\theta = 4\cos^3\left(\frac{\theta}{3}\right) - 3\cos\left(\frac{\theta}{3}\right)$, we simplify equation (23) to become

$$\frac{Q_R}{Q_\pi} = 3 - 4\cos^2\left(\frac{\theta}{3}\right) \tag{22}$$

Since the range of $\theta/3$ is such that $\pi/6 < \theta/3 \leq \pi/4$, the range of $\cos\left(\frac{\theta}{3}\right)$ is such that $(1/2)^{0.5} \leq \cos\left(\frac{\theta}{3}\right) < (3/4)^{0.5}$. Hence, the range of $\frac{Q_R}{Q_\pi}$ is given by

$$0 < \frac{Q_R}{Q_\pi} \leq 1 \tag{23}$$

The above inequalities imply that the optimal Q_π under profit maximization is always greater than or equal to the optimal Q_R under ROII maximization.

Analogous to the above analysis, we can obtain the relative bounds of the optimal demands, prices, and cycle lengths. The results, which can be easily verified, are summarized in the following proposition.

Proposition 1. Given the optimal order quantities, demands, prices, and cycle lengths shown in equations (5) - (8) and (14) - (17), the following relative bounds hold.

a) $0 < \frac{Q_R}{Q_\pi} \leq 1$

b) $\frac{2}{3} < \frac{d_R}{d_\pi} \leq 1$

c) $1 \leq \frac{p_R}{p_\pi} < 1 + \frac{a-c}{3(a+c)}$

d) $0 < \frac{T_R}{T_\pi} \leq 1$

Part b) of Proposition 1 states that d_π is greater than or equal to d_R, but d_π is strictly less than $\frac{3}{2}d_R$. Part c) states that p_π is less than or equal to p_R, but p_R is strictly less than $(1 + \frac{a-c}{3(a+c)})p_\pi$. Finally, Part d) states that T_R is less than or equal to T_π. cf. the analysis in [5] which focuses more on the relative ordering of the optimal decisions and less on the relative bounds of the optimal decisions for logarithmic concave demand functions (which include linear demand functions).

Elasticity Analysis

Let us first investigate the relationship among price, average ordering cost, and price elasticity of demand under profit maximization.

From equation (2),

$$p_\pi - \beta d_\pi = p_\pi(1 - \beta d_\pi/p_\pi)$$
$$= K/Q_\pi + c \qquad (24)$$

The price elasticity of demand at the optimality, $\epsilon_\pi = -p_\pi/(\beta d_\pi) = 1 - (3a/(2(a-c)\cos^2(\frac{\theta}{3})))$ where $\cos\theta = -(\frac{27\beta hK}{4(a-c)^3})^{0.5}$ for $\pi/2 < \theta \leq 3\pi/4$.

Hence, equation (24) becomes $p_\pi(1 + 1/\epsilon_\pi) = K/Q_\pi + c$; i.e.,

$$p_\pi = (\epsilon_\pi/(\epsilon_\pi + 1))(K/Q_\pi + c) \qquad (25)$$

Equation (25) states that the optimal price gets close to the average ordering cost as the demand becomes more elastic with respect to the price. On the other hand, the optimal price gets farther away from the average ordering cost as the demand becomes more inelastic with respect to the price. If we view the term $\epsilon_\pi/(\epsilon_\pi + 1)$ as the markup rate, we can clearly see that as the demand becomes more inelastic (elastic), the markup rate increases (decreases).

Analogous to the profit maximization case above, from equation (12), we obtain the relationship among price, average ordering cost, and price elasticity of demand under ROII maximization as follows.

$$p_R = (\epsilon_R/(\epsilon_R + 1))(K/Q_R + c) \qquad (26)$$

The economic interpretations of equation (26) are similar to those of equation (25) where $\epsilon_R = (c + 2a)/(c - a)$. The relative bounds on the magnitudes of ϵ_π and ϵ_R can be shown to be:

$$\frac{a + c}{2a + c} < \frac{\epsilon_\pi}{\epsilon_R} \leq 1 \qquad (27)$$

Equation (27) states that the demand at the maximum ROII is more elastic than the demand at the maximum profit. This implies that the optimal markup rate for the ROII maximization will be lower than the optimal markup rate for the profit maximization. It can be easily verified that the relative bounds of the magnitudes of the markup rate under ROII maximization, M_R, and the markup rate under profit maximization, M_π, are as follows.

$$1 \leq \frac{M_\pi}{M_R} < \frac{(a+c)(a+2c)}{2c(2a+c)} \qquad (28)$$

The fact that the optimal markup rate under profit maximization is greater than or equal to the optimal markup rate under ROII maximization does not contradict the fact that the optimal price under profit maximization is less than or equal to the optimal price under ROII maximization (See Part c) of Proposition 1). The reason is that the corresponding average ordering cost at the maximum profit is less than or equal to the corresponding average ordering cost at the maximum ROII (See equations (25) and (26)).

Sensitivity Analysis with respect to the Choice of the Objective

In this subsection, we will first analyze the impact on the difference between the optimal order quantities Q_π and Q_R when the inventory holding cost per unit per unit time h changes. From equations (6) and (15), the difference between Q_π and Q_R, $\Delta Q = Q_\pi - Q_R$, is given by

$$\Delta Q = \left(\frac{4K(\alpha-c)}{3\beta h}\right)^{0.5} \cos\left(\frac{\theta}{3}\right) - 3K/(\alpha - c) \tag{29}$$

$$\text{for } \pi/2 < \theta \leq 3\pi/4 \text{ and } \cos\theta = -\left(\frac{27\beta hK}{4(\alpha-c)^3}\right)^{0.5}$$

It can be shown that when $h = 2(\alpha-c)^3/(27\beta K)$ (the highest inventory holding cost per unit per unit time under which the seller is willing to operate; when h is equal to this upper bound, the corresponding optimal profit is zero), $\Delta Q = 0$. It also can be shown that $\frac{\partial \Delta Q}{\partial h} < 0$ for $0 < h \leq 2(\alpha-c)^3/(27\beta K)$. i.e., ΔQ is a monotone decreasing function in h and $\Delta Q|_{h = 2(\alpha-c)^3/(27\beta K)} = 0$. These imply that as h increases, the difference between the optimal order quantity under profit maximization and the optimal order quantity under ROII maximization gets smaller. Furthermore, when $h = 2(\alpha-c)^3/(27\beta K)$, the optimal order quantity under profit maximization is identical to the optimal order quantity under ROII maximization.

Analogous to the above analysis, we can analyze the impact on the difference between the optimal demands d_π and d_R, prices p_π and p_R, and cycle lengths T_π and T_R when the inventory holding cost per unit per unit time h changes. The results, which can be easily verified, are summarized in the following proposition.

Proposition 2.

Given the optimal order quantities, demands, prices, and cycle lengths shown in equations (5) - (8) and (14) - (17), for $0 < h \leq 2(\alpha-c)^3/(27\beta K)$, the following statements hold.

a) The differences between Q_π and Q_R, d_π and d_R, p_π and p_R, and T_π

and T_R all monotonically decrease as h increases.

b) When $h = 2(\alpha - c)^3/(27\beta K)$, the optimal decisions under profit maximization and the optimal decisions under ROII maximization are identical.

The above proposition implies that as h approaches its upper bound of $h = 2(\alpha - c)^3/(27\beta K)$, the optimal decisions on the order quantity, demand, price, and cycle length become less sensitive to the seller's choice of the objective between profit maximization and ROII maximization.

We now proceed to analyze the impact on the difference between the optimal order quantities Q_π and Q_R when the set up cost K changes. Once again, ΔQ (= Q_π - Q_R) is given by equation (29). It can be shown that when $K = 2(\alpha - c)^3/(27\beta h)$ (the highest setup cost under which the seller is willing to operate; when K is equal to this upper bound, the corresponding optimal profit is zero), $\Delta Q = 0$. It also can be shown that as K approaches zero, so do Q_π and Q_R (i.e., ΔQ also approaches zero). Finally, it can be shown that, for $0 < K \leq 2(\alpha - c)^3/(27\beta h)$, ΔQ is a concave function in K and has its maximum value with respect to K when K $\cong 0.1298(4(\alpha - c)^3)/(27\beta h)$. These imply that as K increases or decreases from the critical value of $K \cong 0.1298(4(\alpha - c)^3)/(27\beta h)$, the difference between the optimal order quantity under profit maximization and the optimal order quantity under ROII maximization gets smaller. Furthermore, the optimal order quantity under profit maximization approaches the optimal order quantity under ROII maximization when K approaches zero. Finally, when $K = 2(\alpha - c)^3/(27\beta h)$, the optimal order quantity under profit maximization is identical to the optimal order quantity under ROII maximization.

Analogous to the above analysis, we can analyze the impact on the difference between the optimal demands d_π and d_R, prices p_π and p_R, and cycle lengths T_π and T_R when the setup cost K changes. The results, which can be easily verified, are summarized in the following proposition.

Proposition 3.

Given the optimal order quantities, demands, prices, and cycle lengths shown in equations (5) - (8) and (14) - (17), for $0 < K \leq 2(\alpha-c)^3/(27\beta h)$, the following statements hold.

a) The differences between d_π and d_R as well as p_π and p_R monotonically decrease as K increases.

b) The difference between Q_π and Q_R monotonically decreases as K increases or decreases from the critical value of $K \cong 0.1298(4(\alpha-c)^3)/(27\beta h)$.

c) The difference between T_π and T_R monotonically decreases as K increases or decreases from the critical value of $K \cong 0.1151(4(\alpha-c)^3)/(27\beta h)$.

d) When $K = 2(\alpha-c)^3/(27\beta h)$, the optimal decisions under profit maximization and the optimal decisions under ROII maximization are identical.

e) When K approaches zero, the optimal order quantity and the cycle length under profit maximization approaches the optimal order quantity and the cycle length under ROII maximization.

The above proposition implies that as K approaches its upper bound of $K = 2(\alpha-c)^3/(27\beta h)$, the optimal demand and price become less sensitive to the seller's choice of the objective between profit maximization and ROII maximization. In addition, as K approaches its upper bound of $K = 2(\alpha-c)^3/(27\beta h)$ from the critical value of $K \cong 0.1298(4(\alpha-c)^3)/(27\beta h)$ ($K \cong 0.1151(4(\alpha-c)^3)/(27\beta h)$), the optimal order quantity (cycle length) becomes less sensitive to the the seller's choice of the objective between profit maximization and ROII maximization. Also, as K approaches its lower bound of $K = 0$ from the critical value of $K \cong 0.1298(4(\alpha-c)^3)/(27\beta h)$ ($K \cong 0.1151(4(\alpha-c)^3)/(27\beta h)$), the optimal order quantity (cycle length) becomes less sensitive to the the seller's choice of the objective between profit maximization and ROII maximization.

Finally, by employing similar analysis techniques shown in the cases of changes in h and K, we can analyze the impact on the difference between the optimal order quantity Q_π and Q_R, demands d_π and d_R, prices p_π and p_R, and cycle lengths T_π and T_R when the per unit ordering cost c changes. The results, which can be easily verified, are summarized in the following proposition.

Proposition 4.

Given the optimal order quantities, demands, prices, and cycle lengths shown in equations (5) - (8) and (14) - (17), for $0 < c \leq \alpha - (27\beta hK/2)^{1/3}$, the following statements hold.

a) The differences between Q_π and Q_R, d_π and d_R, and p_π and p_R monotonically decrease as c increases.

b) The difference between T_π and T_R monotonically decreases as c increases or decreases from the critical value of $c \cong \alpha - (245.637\beta hK)^{1/3}$.

c) When $c = \alpha - (27\beta hK/2)^{1/3}$, the optimal decisions under profit maximization and the optimal decisions under ROII maximization are identical.

The above proposition implies that as c approaches its upper bound of $c = \alpha - (27\beta hK/2)^{1/3}$, the optimal order quantity, demand, and price become less sensitive to the seller's choice of the objective between profit maximization and ROII maximization. In addition, as c approaches its upper bound $c = \alpha - (27\beta hK/2)^{1/3}$ from the critical value of $c \cong \alpha - (245.637\beta hK)^{1/3}$, the optimal cycle length becomes less sensitive to the the seller's choice of the objective between profit maximization and ROII maximization.

From Propositions 2 through 4, we summarize that when any of the cost factors among h, K, and c is sufficiently high, the differences between the optimal decisions under profit maximization and the optimal decisions under ROII maximization are negligible. In addition, we note that as a cost factor such as h, K, or c approaches its upper bound, the profit levels π_π and π_R given by equations (9) and (18) approach zero. On the other hand, since $h' < h$ (i.e., there is a positive opportunity cost; an assumption made in the Basic Models section), the ROII levels R_π and R_R given by equations (10) and (19) approach $h/c - h'/c$, which is a strictly positive quantity. Finally, we note that Part b) of Proposition 2, Part d) of Proposition 3, and Part c) of Proposition 4 are consistent with the observation in [5] that, in the case of zero profit, "the profit and ROII models are in agreement on the optimal price-inventory decisions."

ACKNOWLEDGEMENTS

The authors' research is supported in part by a National Science Foundation grant (DDM-9110945). The authors would like to thank Tzong-Yih Chiu for helpful comments.

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PAPER 5.

PRIORITY RATIONING/PRICING AND INTERRUPTION INSURANCE OF ELECTRIC POWER UNDER CUSTOMERS' VALUATION UNCERTAINTY

PRIORITY RATIONING/PRICING AND INTERRUPTION INSURANCE OF ELECTRIC POWER UNDER CUSTOMERS' VALUATION UNCERTAINTY

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ABSTRACT

In this paper, we extend the existing work on the priority rationing of electric power by incorporating commonly shared random factors (such as temperature or humidity) associated with customers' valuation of electric power and the uncertainty associated with the estimation of the total amount of electric power demanded. Next, under the assumption that customers are risk-averse, we formulate an interruption insurance model to transfer the risk of customers to the risk-neutral electric power supplier. Finally, via numerical examples, we attempt to investigate the effects of errors due to the assumptions that customers' valuation and/or the total amount of electric power demanded are constant over time (when they actually vary due to random factors).

INTRODUCTION

The classical theory of electric power priority rationing (see e.g., Wilson [5]) assumes that electric power customers can choose a level of service for each unit of load that will determine its rationing priority in case of shortage. The menu of service options presented to all (potential) customers may be characterized in terms of reliability levels or interruption compensation levels and the corresponding price levels. In either case, however, it is assumed that customers are aware of the power curtailment probability associated with each level of service and self-select the level of service that will maximize their expected net benefits.

In classical priority rationing models (see e.g., Chao and Oren et al. [1]), in case of shortage, the supplier always curtail power in ascending order of interruption loss. In this way, the social loss due to power shortages is minimized and an economically efficient allocation of electric power is achieved. The main task of the supplier is to determine the socially optimal level of reliability for each priority class and the corresponding level of price to be charged, taking the customers' expected net benefit maximizing behavior into consideration.

The cause of electric power shortages in the classical priority rationing models is assumed to be power generation or transmission failures. That is, the interruption losses occur due to physical failure on the supplier side. A more recent priority rationing model by Chao and

Wilson [2], however, incorporates an additional uncertainty on the supplier side. Namely, they mathematically characterize the uncertainty associated with the supplier's spot price of electric power and the corresponding amount of electric power demanded.

In all these previous models of priority rationing, customers' valuation of a unit load of electric power is assumed to be constant over time. However, it is more reasonable to assume that random factors commonly shared by the customers such as temperature or humidity do affect the customers' valuation of a unit load. In addition, the total amount of electric power demanded can be viewed as stochastic because the supplier may not be able to accurately estimate it. In this project, we extend the existing work on the priority rationing by incorporating commonly shared random factors into the customers' valuation of electric power. Also, under the assumption that customers are risk-averse, we formulate an interruption insurance model to transfer the risk of customers to the risk-neutral electric power supplier. Finally, via numerical examples, we investigate the effects of estimation errors due to the assumptions that customers' valuation and/or the total amount of electric power demanded are constant (when they actually vary due to random factors).

PRIORITY RATIONING/PRICING MODEL

We characterize the customer heterogeneity in terms of a customer's valuation index $t \in [0,1]$. This index serves as the preference ranking of a customer relative to other customers in terms of preference for higher reliability of delivering electric power. In this report, larger t corresponds to higher reliability of delivering electric power; and vice versa. Hence, t=0 defines the lowest ranked customer valuation and t=1 the highest. The contingency demand function is characterized by $D(t,\underset{\sim}{F})$, the total amount of electric power demanded in a given period with customer valuation t or higher under contingency $\underset{\sim}{F}$. The vector $\underset{\sim}{F}$ = $(F_1, F_2, F_3,\cdots,F_k)$ denotes a random vector whose elements correspond to estimation factors that affect the customers' valuation distribution of demand. According to our definition of the customer valuation index t, we assume that there is a corresponding utility function $U(t,\underset{\sim}{A})$ for all customer valuation index t under contingency $\underset{\sim}{A}$. The vector $\underset{\sim}{A} = (A_1, A_2, A_3,\cdots, A_h)$ denotes a random vector whose elements corresponding to factors that affect the customers' valuation of utility. Moreover, the contingency supply function is characterized by $S(\underset{\sim}{B})$, the total amount of electric power supplied in a given period under contingency $\underset{\sim}{B}$. The vector $\underset{\sim}{B} = (B_1, B_2, B_3,\cdots,B_j)$ denotes a random vector whose elements correspond to factors that affect the total amount of electric power supplied in a given period. We assume that the sample space γ for $\underset{\sim}{F}$ (α for $\underset{\sim}{A}$, β for $\underset{\sim}{B}$) and the corresponding joint probability distributions

Prob{$\underset{\sim}{f}$} (Prob{$\underset{\sim}{a}$}, Prob{$\underset{\sim}{b}$}) over all possible realizations $\underset{\sim}{f} \in \gamma$ ($\underset{\sim}{a} \in \alpha$, $\underset{\sim}{b} \in \beta$ respectively) are known to the electric power supplier. We generally assume that random factors A, F, and B are independent (cf. the second numerical example in the illustrative numerical example section). Also the electric power supplier is assumed to have complete knowledge of customers' valuation distribution $D(t, \underset{\sim}{F})$ and the form of the utility function $U(t,\underset{\sim}{A})$, but he can not identify the particular type of a customer. Just as in the classical theory of priority rationing of electric power, we will assume that only one interruption may occur per period and the duration of an interruption is constant.

In order to implement this allocation mechanism, discretization schemes for the continuous customer ranks are necessary. For the discretization of customer valuation index t, we employ the concept of customer blocks, or classes. Specifically, we will assume there are M customer blocks and customer block i consists of customer valuation index $t \in [t_i, t_{i-1}]$ where $i = 1, 2, 3,\cdots, M$ and $t_0 = 1$. The customers of type t_i, $i = 0, 1, 2,\cdots M$ will be referred to as boundary customers. The corresponding quantity demanded for customer block i is given by $D(t_i,\underset{\sim}{F}) - D(t_{i-1},\underset{\sim}{F})$ under contingency $\underset{\sim}{F}$ while the boundary customer t_i's utility under contingency $\underset{\sim}{A}$ will be $U(t_i, \underset{\sim}{A})$. Throughout the rest of this report, for notational simplification, we will denote $\underset{\sim}{F}$ as F, $\underset{\sim}{A}$ as A, and $\underset{\sim}{B}$ as B.

We start the priority rationing for electric power as follows : under all possible contingencies, the electric power supplier will deliver electric power to the highest customer block first, until the demand for the first class is met. Only then does the delivery to the second class customers start, and only after the second class, the delivery for the third class, and son on. This rationing scheme terminates when either the supply of electric power is exhausted or when all demands are satisfied.

While the priority rationing rule determines ex-post (i.e., after the electric power generation) the relationship between the priority class and the quantity supplied and demanded, customers' ex-ante (i.e., before the electric power generation) purchase decision will be based on a reliability forecast of that relation. Such a forecast will specify r_i, the delivery reliability of electric power to a customer in priority class i averaged over all possible contingencies. This forecast must take into consideration both the rationing rule and the anticipated response by customers. Such response will obviously depend on the price corresponding to each class, which is controlled by the electric power supplier.

We will now proceed to express the priority rationing rules and reliability level r_i under the priority scheme. For this purpose, we introduce variables denoting the amount of electric power available and the amount of shortage/surplus under each contingency $f \epsilon \gamma$ and $b \epsilon \beta$ as follows :

S_b : electric power supply given $B = b$

Q_{ifb} : remaining demand in class i after using up supply S_b, given $F = f$ and $B = b$

R_{ifb} : remaining supply after delivering class i, given F = f and B = b

$c_{ifb} = (D(t_i,f) - D(t_{i-1},f)) - Q_{ifb}$: actual amount of electric power delivered to class i, given F = f and B = b

The demand for each priority class, given F = f, is given by $D(t_i,f) - D(t_{i-1},f) = D_{if} - D_{i-1f}$ for i = 1, 2, 3,···, M where $D_{0f} = 0$. Consequently, we can express the supply and demand relations with respect to each class under the priority pricing recursively as follows :

$$Q_{ifb} = \text{Max}[(D_{if} - D_{i-1f}) - R_{i-1fb}, 0]$$
$$R_{ifb} = R_{i-1fb} - [(D_{if} - D_{i-1f}) - Q_{ifb}]$$

i = 1, 2, 3,···, M, for all $b \in \beta$ and $f \in \gamma$ where $R_{0fb} = S_b$, $t_0 = 1$, and $D_{0f} = 0$.

According to the priority pricing rule described above, under any given contingency b, the conditional reliability r_{ib} is :

$$r_{ib} = \sum_{f \in \gamma} \text{Prob}\{f\} \frac{c_{ifb}}{D_{if} - D_{i-1f}}$$

From averaging r_{ib} over all possible contingencies, we have

$$r_i = \sum_{b \in \beta} \text{Prob}\{b\} \left[\sum_{f \in \gamma} \text{Prob}\{f\} \frac{c_{ifb}}{D_{if} - D_{i-1f}} \right]$$

Under the proposed scheme, the producer's price schedule will consist of priority prices and the corresponding forecast of delivery reliability for the electric power as shown in Table 1.

We now turn to modeling the customers decisions. We assume that customers are expected value decision makers and the identical price table is provided to all potential customers. Then, each customer t's expected utility and net expected utility when he orders a unit of priority class i are given by,

$$EU_i(t) = r_i \sum_{a \epsilon A} Prob\{a\}\ U(t_i,A)$$

and $$NEU_i(t) = r_i \sum_{a \epsilon A} Prob\{a\}\ U(t_i,A) - p_i.$$

Class	Price	Delivery Reliability of Electric Power
1	P_1	r_1
2	P_2	r_2
⋮	⋮	⋮
M	P_M	r_m

Table 1. Price Table of Priority Pricing for Electric Power

The optimal customers' behavior or self-selection is simple to choose priority level $\hat{i}$, where $NEU_{\hat{i}} = \underset{i}{Max}\ NEU_i(t)$. We represent the market segmentation of all customers in terms of the following boundary customers relations, given appropriate prices, p_1, p_2, p_3,$\cdots$, p_M.

For i = 1, 2, 3,$\cdots$, M-1

$$r_i \sum_{a \epsilon A} Prob\{a\}\ U(t_i,A) - p_i = r_{i+1} \sum_{a \epsilon A} Prob\{a\}\ U(t_i,A) - p_{i+1}$$

$$r_M \sum_{a \epsilon A} Prob\{a\}\ U(t_M,A) - p_M = 0.$$

The above relations state that the boundary customer t_i, i = 1, 2, 3,$\cdots$, M-1, is indifferent between purchasing priority class i and i+1 and the last boundary customer t_M is indifferent between subscribing to priority level M or withdrawing from the market. Also it can be easily verified that all non-boundary customers of customer valuation index $t \in [t_i, t_{i-1}]$ will purchase priority class i, i = 1, 2, 3,⋯, M.

For the basic model described thus far, the corresponding formulae for the expected profit, expected customer surplus, and expected total surplus are obtained as follows.

Expected Profit :

$$E\pi = \sum_{f \in \gamma} \text{Prob}\{f\}[\sum_{i=1}^{M} p_i(D(t_i,f) - D(t_{i-1},f))]$$

Expected Customer Surplus :

$$ECS = \sum_{f \in \gamma} \text{Prob}\{f\}\left[\sum_{a \in A} \text{Prob}\{a\}[\sum_{i=1}^{M}\int_{t_{i-1}}^{t_i} r_i U(t,A)dD(t,F)]\right] - \pi$$

Expected Total Surplus :

$$ETS = \sum_{f \in \gamma} \text{Prob}\{f\}\left[\sum_{a \in A} \text{Prob}\{a\}[\sum_{i=1}^{M}\int_{t_{i-1}}^{t_i} r_i U(t,A)dD(t,F)]\right]$$

So far, the entire formulation for the electric power expected total surplus maximization problem is shown as follows.

<u>The Formulation for the Electric Power Expected Total Surplus Maximization Problem</u>

$$\text{Maximize ETS} = \sum_{f \in \gamma} \text{Prob}\{f\}\left[\sum_{a \in A} \text{Prob}\{a\}[\sum_{i=1}^{M}\int_{t_{i-1}}^{t_i} r_i U(t,A)dD(t,F)]\right]$$

subject to :

$$1 = t_0 \geq t_1 \geq t_2 \geq \cdots \geq t_M$$

$$p_1 \geq p_2 \geq p_3 \geq \cdots\cdots \geq p_M$$

all variables ≥ 0

Boundary Customers Relations :

$$r_i \sum_{a \epsilon A} \text{Prob}\{a\}\ U(t_i, A) - p_i = r_{i+1} \sum_{a \epsilon A} \text{Prob}\{a\}\ U(t_i, A) - p_{i+1}$$

$$r_M \sum_{a \epsilon A} \text{Prob}\{a\}\ U(t_M, A) - p_M = 0.$$

Priority Rationing Relations :

$$Q_{ifb} = \text{Max}[(D_{if} - D_{i-1f}) - R_{i-1fb},\ 0]$$

$$R_{ifb} = R_{i-1fb} - [(D_{if} - D_{i-1f}) - Q_{ifb}]$$

$i = 1, 2, 3, \cdots, M$, for all $b \epsilon \beta$ and $f \epsilon \gamma$

where $R_{0fb} = S_b$, $t_0 = 1$, and $D_{0f} = 0$.

Reliability of Delivery Relations :

$$r_i = \sum_{b \epsilon \beta} \text{Prob}\{b\} \left[\sum_{f \epsilon \gamma} \text{Prob}\{f\} \frac{c_{ifb}}{D_{if} - D_{i-1f}} \right]$$

INTERRUPTION INSURANCE MODEL

In the literature of electric power demand management, there have been numerous articles on insurance for power interruption. In this section, we will explore a way to implement the insurance under the assumptions of random factors (e.g., temperature or humidity) in customers' valuation and the uncertainty of the total amount of electric power demanded. Customers are assumed to be risk averse expected utility maximizers (see e.g., Varian [4] or Oren and Doucet [3]) and the electric power supplier is assumed to be risk-neutral. To quote from Wilson [5], "If customers are risk averse, then full efficiency requires that risks are shared efficiently among the customers and the firm. In important application such as power, a state enterprise or public utility is much less risk averse than each customer. Consequently, we investigate the case that the firm or a private underwriter offers compensatory insurance against the risk of loss from service interruptions, and does so at actuarially fair rates". An identical premium price schedule is offered to all potential customers. The proposed premium price schedule consists of a service charge s paid only when electric power is delivered, an insurance premium G_i, a compensation level K_i and the corresponding forecast of delivery reliability r_i of electric power. The proposed tariff is shown as table 2.

Under this premium price schedule, a consumer t selecting class i will receive a net benefit of $EU_i(t) - s - G_i$ with probability r_i and $K_i - G_i$ with probability $1 - r_i$. To quote from Varian [4], "if the customer is a risk averse expected utility maximizer, and if he is offered fair insurance against a loss, then he will optimally choose to fully insure". Therefore, for a risk-averse customer, he will gladly choose full insurance to avoid risk. The full insurance relation is shown as follows.

$$EU_i(t) - s - G_i = K_i - G_i \text{ for } i = 1,2,3,\cdots M.$$

Class	Reliability	Price		Compensation
		Service charge	Premium	
1	r_1	s	G_1	K_M
2	r_2	for all	G_2	K_M
⋮	⋮	priority classes	⋮	
M	r_M		G_M	K_M

Table 2. Tariff for interruption insurance model

Under the proposed interruption insurance scheme, the boundary customer relations should be modified as follows.

$$r_i[\sum_{a\epsilon A} \text{Prob}\{a\}\ U(t_i,A) - s] + (1 - r_i)\ K_i - G_i$$

$$= r_{i+1}[\sum_{a\epsilon A} \text{Prob}\{a\}\ U(t_i,A) - s] + (1 - r_{i+1})\ K_{i+1} - G_{i+1}$$

$$r_M[\sum_{a\epsilon A} \text{Prob}\{a\}\ U(t_M,A) - s\ ] + (1 - r_M)\ K_M - G_M = 0.$$

The corresponding formulae for the expected profit, expected customer surplus, and expected total surplus are obtained as follows.

Expected Profit :

$$E\pi = \sum_{f\epsilon\gamma} \text{Prob}\{f\}[\sum_{i=1}^{M}(r_i s+G_i)(D(t_i,f) - D(t_{i-1},f))]$$

$$- \sum_{f\epsilon\gamma} \text{Prob}\{f\}[\sum_{i=1}^{M}(1-r_i)K_i(D(t_i,f) - D(t_{i-1},f))]$$

Expected Customer Surplus :

$$ECS = \sum_{f\epsilon\gamma} \text{Prob}\{f\}\left[\sum_{a\epsilon A} \text{Prob}\{a\}[\sum_{i=1}^{M}\int_{t_{i-1}}^{t_i} [r_i U(t,A)+(1-r_i)K_i]dD(t,F)]\right]$$

$$- \sum_{f\epsilon\gamma} \text{Prob}\{f\}[\sum_{i=1}^{M}(r_i s+G_i)(D(t_i,f) - D(t_{i-1},f))]$$

Expected Total Surplus :

$$ETS = \sum_{f\epsilon\gamma} \text{Prob}\{f\}\left[\sum_{a\epsilon A} \text{Prob}\{a\}[\sum_{i=1}^{M}\int_{t_{i-1}}^{t_i} r_i U(t,A)dD(t,F)]\right]$$

So far, the entire formulation for the interruption insurance model is shown as follows.

The Formulation for the Interruption Insurance Model

$$\text{Maximize ETS} = \sum_{f\epsilon\gamma} \text{Prob}\{f\}\left[\sum_{a\epsilon A} \text{Prob}\{a\}[\sum_{i=1}^{M}\int_{t_{i-1}}^{t_i} r_i U(t,A)dD(t,F)]\right]$$

subject to :

$$1 = t_0 \geq t_1 \geq t_2 \geq \cdots \geq t_M$$

$$G_1 \geq G_2 \geq G_3 \geq \cdots\cdots \geq G_M$$

$$K_1 \geq K_2 \geq K_3 \geq \cdots\cdots \geq K_M$$

all variables ≥ 0

Full Insurance Relations :

$$EU_i(t) - s - G_i = K_i - G_i \text{ for } i = 1,2,3,\cdots M.$$

Boundary Customers Relations :

$$r_i[\sum_{a\epsilon A} \text{Prob}\{a\}\ U(t_i,A) - s] + (1 - r_i)\ K_i - G_i$$

$$= r_{i+1}[\sum_{a\epsilon A} \text{Prob}\{a\}\ U(t_i,A) - s] + (1 - r_{i+1})\ K_{i+1} - G_{i+1}$$

$$r_M[\sum_{a\epsilon A} \text{Prob}\{a\}\ U(t_M,A) - s\ ] + (1 - r_M)\ K_M - G_M = 0.$$

Priority Rationing Relations :

$$Q_{ifb} = \text{Max}[(D_{if} - D_{i-1f}) - R_{i-1fb},\ 0]$$

$$R_{ifb} = R_{i-1fb} - [(D_{if} - D_{i-1f}) - Q_{ifb}]$$

$i = 1, 2, 3, \cdots, M$, for all $b \epsilon \beta$ and $f \epsilon \gamma$

where $R_{0fb} = S_b$, $t_0 = 1$, and $D_{0f} = 0$.

Reliability of Delivery Relations :

$$r_i = \sum_{b\epsilon\beta} \text{Prob}\{b\} \left[\sum_{f\epsilon\gamma} \text{Prob}\{f\} \frac{c_{ifb}}{D_{if} - D_{i-1f}}\right]$$

By comparing the interruption insurance model in this section and the priority rationing/pricing model in the previous section, we have the following observations.

1) If we set the compensation levels for all classes equal to zero (e.g., the electric power supplier will not offer the interruption insurance service), then the interruption insurance model will be reduced to the priority rationing/pricing model and the relation of the price in the priority rationing/pricing model and service charge and insurance premium in the interruption insurance is $P_i = r_i s + G_i$.

2) If the service charge and the compensation level is restricted to be zero, then there is no difference between priority/pricing model and interruption insurance model, and the price P_i of priority/pricing model equals to the insurance premium G_i of interruption insurance model.

ILLUSTRATIVE NUMERICAL EXAMPLES

In this section, we will discuss three numerical examples. In the first (second) example, we will investigate the effects of erroneously assuming that the demand (the customers' valuation) for the electric power is constant over time when it actually is not. Finally, in the third example, we illustrate how the interruption insurance scheme can be implemented under the assumptions of random factors (e.g., temperature or humidity) in customers' valuation and the uncertainty of the total amount of electric power demanded.

Different demand function assumptions

In this subsection, we discuss two numerical examples under different demand function assumptions. For the first model, we assume that the demand function $D(t,\tilde{F})$ contains random factors $\tilde{F}$. For the second model, we assume that the demand function is constant. Specifically, in order to investigate the effects of erroneously assuming that the demand for the electric power is constant over time, we will assume that the demand function of the second model is the expected value of the first model.i.e., $ED(t) = \sum_{f \epsilon \gamma} \text{Prob}\{f\}\, D(t,\tilde{F})$. The relevant utility, demand, and supply functions are assumed to be as follows:

$$U(t,A) = t^{1/A} \qquad \text{if } A=1 \text{ with probability} = 0.25$$
$$\qquad\qquad\qquad \text{if } A=2 \text{ with probability} = 0.75$$

$$D(t,F) = 1 - t^F \qquad \text{if } F=1 \text{ with probability} = 0.25$$
$$\qquad\qquad\qquad\quad \text{if } F=2 \text{ with probability} = 0.75$$
$$S(B) = 0.9 - 0.1B^2 \qquad \text{if } B=1 \text{ with probability} = 0.5$$
$$\qquad\qquad\qquad\quad \text{if } B=2 \text{ with probability} = 0.5$$

where $t \in [0,1]$.

We also assume that the number of priority class $M = 2$. We solve this problem employing the formulation shown in the priority rationing/pricing model section. The resulting optimal solution is given by Table 3.

The resulting expected market share (i.e., expected demand of electric power), expected profit, expected consumer surplus, and expected total surplus from classes 1 and 2 are summarized in table 4.

Let us now suppose that the demand for electric power is assumed to be constant over time and the corresponding demand function is given by $D(t) = 1 - \frac{1}{4} t - \frac{3}{4} t^{0.5}$ (i.e., the expected demand function of the first model). Under this assumption, the resulting optimal solution is given by Table 5.

The corresponding expected market share, expected profit, expected consumer surplus, and expected total surplus from classes 1 and 2 are summarized in table 6.

Class	Price	Reliability of Delivering Electric Power
1	$P_1 = 0.516$	$r_1 = 0.875$
2	$P_2 = 0.083$	$r_2 = 0.214$

Table 3. Price Table under demand function with random factors

Class	EMS	$E\pi$	ECS	ETS
1	0.687	0.354	0.157	0.511
2	0.232	0.019	0.008	0.027

Table 4. Welfare outcomes with demand uncertainty

(EMS denotes the expected market share)

Class	Price	Reliability of Delivering Electric Power
1	$P_1 = 0.666$	$r_1 = 1.000$
2	$P_2 = 0.277$	$r_2 = 0.500$

Table 5. Price Table under expected demand function

Class	EMS	$E\pi$	ECS	ETS
1	0.500	0.333	0.116	0.449
2	0.300	0.083	0.019	0.102

Table 6. Welfare outcomes with constant demand function

From Tables 3-6, we observe the followings :

1) The corresponding levels of expected total surplus under these two models are 0.538 and 0.551 respectively. Hence, the second model overestimates the level of expected total surplus by 2.41%.
2) The prices as well as the corresponding reliability levels for priority classes 1 and 2 have increased in the second model.
3) The expected market shares, the expected profit levels, the expected customer surplus levels as well as the expected total surplus levels for priority class 1 (priority class 2) in the first model are larger (smaller) than those in the second model.

As shown by 1), 2), and 3), the constant demand function assumption may substantially distort the critical economic quantities such as the levels of reliability, the corresponding prices, and the total surplus levels.

Different utility function assumptions

In this subsection, we investigate two numerical examples under different utility function assumptions. In this particular case, we will assume that F is a function of A. Specifically, F = A. For the first model, we assume that the utility function $U(t,\underset{\sim}{A})$ contains commonly shared random factors $\underset{\sim}{A}$. For the second model, in order to investigate the effects of erroneously assuming that the customers' valuation is constant over time, we will take the expectation of utility function of the first model over all contingency $\underset{\sim}{A}$, i.e., $EU(t) = \sum_{a \epsilon A} Prob\{a\}\ U(t,\underset{\sim}{A})$, as the (constant) utility function. The relevant utility, demand, and supply functions are assumed to be as follows :

$$U(t,A) = t \times \exp(A) \quad \text{if } A=1 \text{ with probability} = 0.25$$
$$\qquad\qquad\qquad\qquad \text{if } A=10 \text{ with probability} = 0.75$$
$$D(t,A) = 1-t^{A} \quad \text{if } A=1 \text{ with probability} = 0.25$$
$$\qquad\qquad\qquad\qquad \text{if } A=10 \text{ with probability} = 0.75$$
$$S(B) = 0.9 - 0.1B^{2} \quad \text{if } B=1 \text{ with probability} = 0.5$$
$$\qquad\qquad\qquad\qquad \text{if } B=2 \text{ with probability} = 0.5$$
$$\text{where } t \in [0,1].$$

We also assume that the number of priority class $M = 2$. We solve this problem employing the formulation shown in the priority rationing/pricing model section. The resulting optimal solution is given by Table 7.

The resulting expected market share (i.e., expected demand of electric power), expected profit, expected consumer surplus, and expected total surplus from classes 1 and 2 are summarized in table 8.

Let us now suppose that the customers' valuation is assumed to be constant over time and the corresponding utility function is given by $U(t) = \frac{1}{4}\, t \times \exp(1) + \frac{3}{4}\, t \times \exp(10)$ (i.e., the expected utility function of the first model). Under this assumption, the resulting optimal solution is given by Table 9.

The corresponding expected market share, expected profit, expected consumer surplus, and expected total surplus from classes 1 and 2 are summarized in table 10.

Class	Price	Reliability of Delivering Electric Power
1	P_1 = 10635.2	r_1 = 0.859
2	P_2 = 2065.07	r_2 = 0.250

Table 7. Price Table under utility function with random factors

Class	EMS	$E\pi$	ECS	ETS
1	0.637	6776.68	3932.03	10708.71
2	0.237	489.632	147.848	637.48

Table 8. Welfare outcomes with customers' valuation uncertainty (EMS denotes the expected market share)

Class	Price	Reliability of Delivering Electric Power
1	P_1 = 5476.36	r_1 = 0.738
2	P_2 = 413.013	r_2 = 0.125

Table 9. Price Table under expected utility function

Class	EMS	$E\pi$	ECS	ETS
1	0.874	4786.34	4665.445	9451.78
2	0.076	31.389	23.50645	54.895

Table 10. Welfare outcomes with constant customers' valuation

From Tables 7-10, we observe the following:

1) The corresponding levels of expected total surplus under these two models are 11346.192 and 9506.4 respectively. Hence, the second model underestimates the level of expected total surplus by 19.35%.
2) The prices as well as the corresponding reliability levels for priority classes 1 and 2 have decreased in the second model.
3) The expected market shares as well as the expected customers' surplus for priority class 1 (priority class 2) in the first model are smaller (larger) than those in the second model.
4) The expected profit levels as well as the expected total surplus levels for priority class 1 and 2 have decreased in the second model.

As shown by 1), 2), 3), and 4), the constant customers' valuation assumption may substantially distort the critical economic quantities such as the levels of reliability, the corresponding prices, and the total surplus levels.

A Numerical Example for Interruption Insurance Model

In this subsection, we employ the relevant utility, demand, and supply functions from the first set of numerical examples and consider both demand function and utility function with randomness. We solve this problem by employing the formulation shown in the interruption insurance model. The resulting optimal solution is given by Table 11.

The corresponding expected market share, expected profit, expected consumer surplus, and expected total surplus from classes 1 and 2 are summarized in table 12 .

Class	Reliability	Price		Compensation
		Service charge	Premium	
1	$r_1 = 0.875$	$s = 0.300$	$G_1 = 0.297$	$K_1 = 0.355$
2	$r_2 = 0.214$		$G_2 = 0.085$	$K_2 = 0.085$

Table 11. Price table for interruption insurance model

Class	EMS	$E\pi$	ECS	ETS
1	0.687	0.354	0.157	0.511
2	0.232	0.019	0.008	0.027

Table 12. Welfare outcomes for interruption insurance model

CONCLUDING REMARKS

In this paper, we extended the existing work on the priority rationing by incorporating the commonly shared random factors into the customers' valuation of electric power and the estimation uncertainty into the total amount of electric power demanded. Moreover,under the assumption that customers are risk-averse, we formulate an interruption insurance model to transfer the risk of customers to the risk-neutral electric power supplier. We also attempted to investigate the effects of errors due to the assumptions that customers' valuation and/ or the total amount of electric power demanded are constant over time (when they actually vary due to random factors) via numerical examples.

The model presented in this paper as well as the previous models of priority rationing can be further improved by considering uncertainty associated with customers' quantity demanded. In contrast to the demand uncertainty due to the supplier's inability to estimate the correct quantity demanded, there is additional variations in the total quantity demanded due to customers' changes in the optimal quantity of electric power to consume. How these additional variations will affect the levels of reliability, the corresponding prices, and the level of total surplus is an important issue for further research.

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GENERAL CONCLUSION

In this study, we first extended Min (1992b) by designing an alternative model under the Bertrand behavioral assumption and by performing sensitivity analysis on both the Cournot and Bertrand models. Interesting economic implications regarding critical elements of EOQ models such as the setup and inventory holding costs as well as the critical elements of microeconomic market theory such as the market price and the number of competing producers have been derived from the equilibrium conditions and subsequent sensitivity analyses. Next, we allowed the options of investing in reducing the setup and inventory holding costs are available to the producers and presented a unique insight as to why several Japanese and American producers are striving to reduce the setup costs under ever increasing competition. Specifically, it has shown that, for a profit maximizing producer, as the number of competing producers increases, his optimal strategy dictates that he reduce his setup and inventory holding costs.

The EOQ model developed in the first two papers are applicable for broad classes of convex cost function and concave inverse demand function. Our models relate to general practices since numerous industries and firms apply EOQ based decision making under competition. There are several possible extensions that will further improve the relevance of our models to general practices. They include incorporation of more sophisticated features such as quantity discount price schedules, finite production rates, shortages, delivery lags, and promotional (e.g., advertising) effects as well as stochastic demand rates. From the perspective of Zero Inventory Philosophy, it would be of interest to study the effects of competition on process quality improvement and effective capacity in conjunction with the setup cost reduction (see e.g., Porteus(1986) and Spence and Porteus(1987)).

From the perspective of game theory, both Cournot model shown in the first two papers and Bertrand model shown in the first paper can be considered as only an initial step toward better understanding of competitive inventory policies. It is our hope that more sophisticated equilibrium concepts of game theory (e.g., subgame perfect equilibrium for sequential decisions) will be exploited in the future research on the competitive inventory policies.

In the third paper, we have shown how to formulate the profit maximization problem for intermediary firms utilizing the special structure of the firms' environments. The optimal selling quantity and purchasing price are derived in terms of fixed and variable selling costs, supply proportionality constant, annual holding cost, selling price, inspection cost, and defect-rate.

The observation that the price between producers and intermediary firms is determined by intermediary firms. From the aspects of producers, in order to operate efficiently, they must take the relation between the supply rate of inputs and their corresponding prices into account. For such firms, the model in the third paper can be a basis for further research.

In the fourth paper, for a single seller, we compared and contrasted the optimal inventory and pricing policies under profit maximization vs. ROII maximization when demand is linear in price. Specifically, we have shown that when a cost factor such as the setup cost, inventory holding cost per unit per unit time, or per unit ordering cost after the setup is sufficiently high, the choice of the objective between profit maximization and ROII maximization is inconsequential to the seller in so far as his optimal decisions are concerned.

In the fifth paper, we extended the existing work on the priority rationing by incorporating the commonly shared random factors into the customers' valuation of electric power and the estimation uncertainty into the total amount of electric power demanded. Moreover, under the assumption that customers are risk-averse, we formulate an interruption insurance model to transfer the risk of customers to the risk-neutral electric power supplier.

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