HEAT TRANSFER USING THE
LATENT HEAT OF FUSION

by

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INTRODUCTION

The latent heat of fusion, hereafter referred to as latent heat, is defined in the Handbook of Chemistry and Physics (8, p. 3097) as "The quantity of heat necessary to change one gram of solid to a liquid with no temperature change." To gain further insight into the use of the latent heat consider the simple example of ice used for cooling. Ice absorbs heat from its surroundings until it reaches its melting temperature. Then the ice continues to absorb heat without temperature change while it melts. Finally, the melted ice begins to heat up again. Two important factors in using the latent heat are seen in this example: (1) There must be a heat source with a temperature equal to or higher than the melting temperature of the solid to be melted. (2) No temperature change takes place in the material while melting takes place, but thermal energy is absorbed. The purpose of this thesis is to investigate one application of this physical property.

The question one might well ask at this point is how does this form of thermal energy absorption compare with other commonly used methods. The three most important effects of thermal energy absorption are the temperature increase of a fluid, vaporization of a material and melting of a material. For the purpose of comparison, the thermal energy absorbed by one gram of material by a particular method of heat absorption is used. For the thermal energy absorbed by a material the change in termerature will be arbitrarily defined as the difference between the melting point and boiling point of the material. This change in temperature will be excessively large in some instances. Therefore another comparative value of 100 °C will be used. Thus for comparison the thermal energy absorbed by melting one gram of material becomes

$$\triangle H_M = L \quad (1)$$

where L is the latent heat in calories per gram. For a change in temperature

$$\triangle H_{\triangle T} = C \triangle T \quad (2)$$

$$\triangle H_{100} = 100C \quad (3)$$

where C is the specific heat in calories per gram-degree centigrade and $\triangle T$ is the change in temperature as defined above in degrees centigrade. For vaporization

$$\triangle H_V = L_V$$

where L_V is the latent heat of vaporization in calories per gram. The quantity $\triangle H$ in each case represents the change in enthalpy per gram of material.

Another useful quantity which will be discussed in more detail later is the thermal energy absorbed per unit volume which is designated by $\triangle H\rho$. The quantity $\triangle H$ will be defined as Equation 1, 2, or 3 and ρ will be the density of the material being discussed. Values for these quanti-

Table 1. Thermal energy absorption.

	ΔT °C	$\Delta H_{\Delta T}$ cal/g	ΔH_M cal/g	ΔH_v cal/g	ΔH_{100} cal/g	$\Delta H_{\Delta T}\rho$ cal/cm^3	$\Delta H_M\rho$ cal/cm^3	$\Delta H_v\rho$ cal/cm^3	$\Delta H_{100}\rho$ cal/cm^3
Water, H_2O	100-0	100.	79.71	539.55	100.	100	79.71	539.55	100
Sodium, Na	880-97.5	250.4	31.7		32.	232.5	30.18		
Sodium Chloride, NaCL	1465-804.3		124.		21.7				
Sodium Fluoride NaF	1704-992.2		186.		27.9				
Lithium Nitrate, $LiNO_3$	(250)		88.5		39.				
Potassium Fluoride, KF	1502-859.9		108.		22.5				
Aluminum, AL	2057-659.7	349.3	76.8		25.0	832.1	20.73		59.55
Nickel, Ni	2900-1455	187.8	73.8		13.		634.7		
Mercury, H_g	356.58+39	12.78	2.82	70.6	3.23	173.0	40.02	956.0	43.73
Phosphorus, P	287-44.2	52.7	5.03	130.	21.7	123.3		304.2	50.78
Sulfuric Acid H_2SO_4	326-10.35	107.0	24.0	122.1	33.9				
Bismuth, Bi	1560-271.3	37.63	12.64		2.92	385.3	122.2		29.9
Lead, Pb	1620-327.43	48.47	5.86		3.75	516.0	64.75		39.92
Lithium, Li	1336-186	1580.1			137.4	843.77			73.37

ties are tabulated for several elements and inorganic compounds in Table 1. These values in the table are not exact, since "typical" values of the physical properties were used. Using the values in Table 1, it can be noted that the heat absorption by melting is at least comparable to the other processes except in the case of the vaporization of water. In most cases the heat absorption in the melting process is approximately equivalent to the other processes.

Review of the Literature

Altman et al. (1) discussed the possibility of using the latent heat of fusion in an energy storage device. In this article the authors discussed such a device in general and perform calculations for a simplified case to be used in conjunction with a solar mirror in satellites. No other applications of this principle have been found. The lack of literature on this subject of using the latent heat indicates the need for more research in the area.

In carrying out the heat transfer analysis use was made of Glasstone and Sesonske (5). During the investigation of the melting problem several approaches were reviewed including those presented by Carslaw and Jaeger (3), Goodman (6), Goodman and Shea (7), Ingersoll and Zobel (12), Landau (13), Murray and Landis (16), Ross (19), and Stolz (20). The method chosen was one of two methods presented by Murray and Landis

(16). The analytic solution of heat conduction in a composite sphere given by Carslaw and Jaeger (3) is also referenced. An experimental study of freezing and melting is given by Thomas and Westwater (21).

Use was made of Higdon et al. (10), Murphy (15), Roark (18), and Timoshenko and Goodier (22) in the stress analysis of spherical shells. Handbook of Chemistry and Physics (8), Handbook of Thermophysical Properties of Solid Materials (9), Hultgren (11), and Metal Handbook (14) were used as sources for the material properties used in Table 1 and Appendix A.

Possible Uses of the Latent Heat

There are two possible modes of application of the energy absorption power of the latent heat. In the first, or "one shot" mode, the material acts as a sort of safety or protective device. After the material melts, it is no longer of any use so far as this method is concerned. The material must be replaced or cooled in some manner before the device is again ready for service. One possible example of this mode is that the latent heat of some material might be used to keep a shipping container below a given temperature during an accidental fire. This same principle might be useful in protecting instruments from high transient temperatures. Another possible application is to use the latent heat of a material to absorb the heat during shutdown of high temperature devices.

The second mode is one of continuous operation. In this mode the latent heat of a material is used to transfer the heat away from a continuous heat source. This is the mode to which the bulk of this paper will be devoted.

The most common method of cooling or obtaining heat from a heat source is to circulate a coolant through tubes placed through a heat source, and then pass the coolant through a heat exchanger to extract the thermal energy. This is the method commonly used to obtain thermal energy from nuclear power reactors. This thermal energy is then used to generate high pressure steam which in turn drives steam turbine generators. The coolants used vary widely, but a few of the most commonly used coolants are water, liquid metals, organic fluids, and gases. Another important method of using the thermal energy is by turning the water directly into steam by boiling the water in the reactor.

One alternate method might be to allow bar stocks of a coolant material to contact the heat source. The liquid, containing the thermal energy in the form of latent heat, could be drained off. The heat energy could then be released in a heat exchanger as the material is resolidified into bar stock again. This might be improved upon by allowing the bar stock to be immersed in the molten material which is in contact with the heat source. The transport of the bar stock or bulk material and the method of reformation would most certainly present formidable engineering problems.

A more favorable alternative would be to use a slurry with the particles of coolant material suspended in a liquid or gas. The biggest difficulty would be to maintain suspension. There would also be problems during shutdown and startup. Choosing coolant materials and transport media which would lend themselves to formation of suspension for both phases of the coolant material as well as having satisfactory thermodynamic and heat transfer properties would be a formidable, if not impossible, task. However, if there were ways to circumvent this major difficulty, this alternative could appear to be somewhat more attractive than other methods now used.

Another possibility is to use this suspension concept to increase the efficiency of present cooling system designs. Finely divided particles could possibly be added to present systems to increase the heat absorption capacity of the coolant stream without increasing the required temperature.

THE HEAT TRANSFER SYSTEM

The method proposed to overcome the difficulties listed in the preceding section is to enclose the coolant material in a shell. These particles could be either used to form a suspension or to be mechanically transported through the coolant loop. The diagram for a system incorporating this idea is given in Figure 1. It is similar to many other coolant loops used for heat transfer. Probably the most efficient plan would be to fill the system with a liquid metal of high thermal conductivity and allow fluid pressure to force the suspended particles around the coolant loop.

Starting at Point 1 in Figure 1, the particles would go through the reactor. The heat energy produced in the reactor would melt the coolant material (but not the particle shells). Then the particles would pass into a heat exchanger where the heat energy in the particles would be given up and the coolant material in turn solidifies. Then the particles would be ready to start the cycle over again. The pump is necessary to maintain proper coolant flow through the reactor.

The primary advantage of this alternative is that it combines continuous operation with a relatively small overall temperature change across the reactor. In addition, there are no large density changes like those which occur in vaporizing water to steam. The use of a shell concept allows

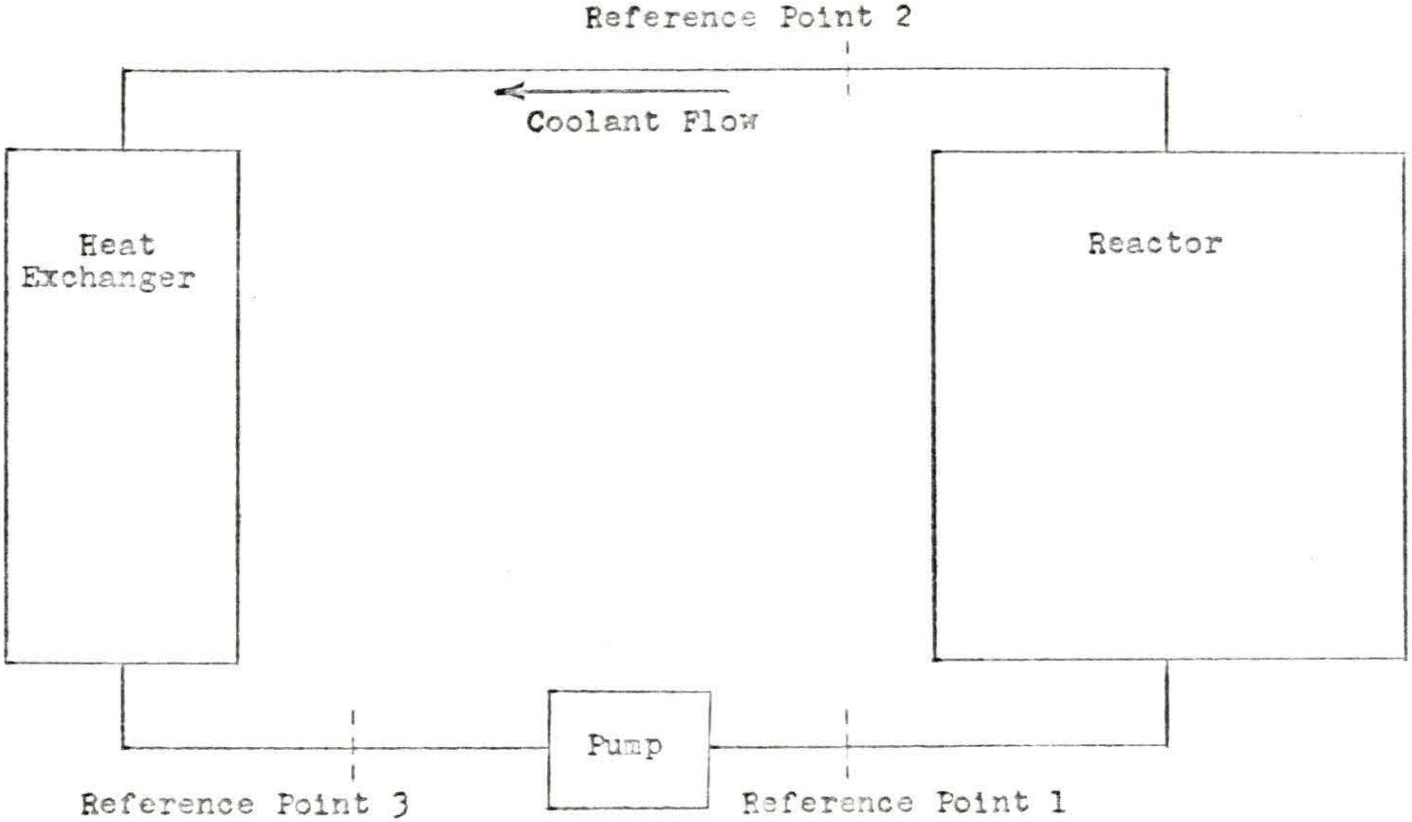


Figure 1. The heat transfer system block diagram.

a wide choice of coolant materials. In many applications these items would not offer any particular advantage, but in a nuclear reactor the properties of the reactor are significantly affected by changes in temperature and density. One interesting potential for a system of this type is the design of a nearly homogeneous nuclear reactor with continuous fuel recycling.

These advantages, however, are not to be gained without the addition of some difficulties. The most prominent disadvantage is in the fabrication of large numbers of coolant particles. The fabrication of the small coolant particles could be difficult and therefore costly. Also there may be larger pumping power requirements than the more conventional systems because of the need to pump both the coolant and transport medium. Still another difficulty may be the erosion effects of the particles inside the coolant tubes.

It appears the final overall judgment of the system will be economic. The increased cost of fabrication and pumping power must be outweighed by the savings created by the more even temperature distributions. These savings would come about mainly through more even fuel burnup and reduced fuel fabrication costs.

In order to gain a better understanding of the proposed system given in Figure 1, it will be postulated to be in steady state operation with heat energy QR being produced in the reactor per unit time. It is further assumed that the heat energy losses are negligible, and that any friction losses P_f, are given by

$$P_p = P_f \tag{5}$$

where P_p is the pumping power.

Define the following by:

L = latent heat of fusion for the coolant material

T_{melt} = melting temperature of the coolant material

η = weight fraction of the coolant material to the total material transported

Q_E = energy removed from the coolant stream by the heat exchanger per unit time

P_m = density of the coolant material which is assumed constant

Assume that the coolant particles are small and well mixed with the transport medium so that the coolant stream can be considered homogeneous with density ρ_c. The coolant flow will have a constant mass flow rate, w. Then using the first law of thermodynamics

$$\text{Change in energy} = \text{heat added} + \text{work done} = 0$$

or

$$(Q_R - Q_E)t = (P_f - P_p)t \tag{6}$$

where t is some specified increment of time. The right hand side of Equation 6 is zero therefore

$$Q_R = Q_E \tag{7}$$

Since the main interest is in the effect of the latent heat, the mean coolant stream temperature will be taken as constant at the coolant material melting temperature for the points noted in Figure 1. Then, if ξ_i is the weight fraction of the coolant material which is melted at the i^{th} point,

$$Q_R = w\eta(\xi_2 - \xi_1)L \qquad (8)$$

where 1 and 2 refer to the points 1 and 2 in Figure 1. A similar expression may be written for the energy removed by the heat exchanger that is

$$Q_E = w\eta(\xi_2 - \xi_3)L \qquad (9)$$

Since heat energy is neither added nor subtracted in the pump, then

$$\xi_1 = \xi_3 \qquad (10)$$

This also follows from Equations 7, 8, and 9.

The reactor is postulated to contain n coolant flow tubes one of which is shown in Figure 2. The heat transfer expressions for similar coolant tubes and conventional coolants, using the specific heat of a fluid coolant, are given in Glasstone and Sesonske (5, p. 364).

The differential mass of coolant material dm which passes along the differential length of tube dx per unit time is given by

$$dm = 2\pi R_t v \rho_c \qquad (11)$$

From Figure 2 and the previous definitions, it is seen that

$$w = nv\, 2\pi R_t \rho_c \qquad (12)$$

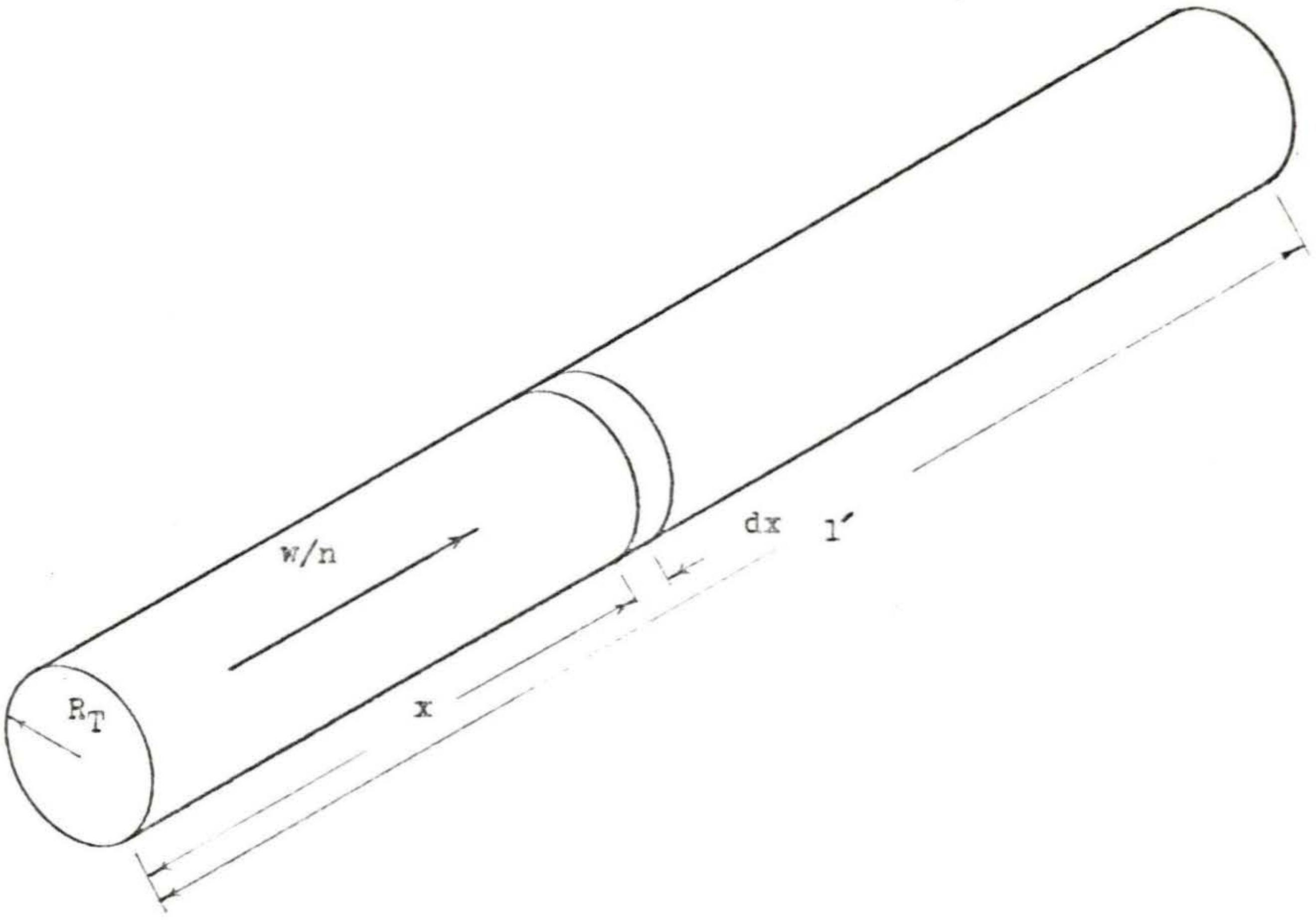


Figure 2. Typical coolant tube.

where v is used here as the mean speed of the coolant stream.

The rate dq(x) at which heat is added to the stream in a differential length dx for the i^{th} tube is given by

$$\begin{aligned} dq_i(x) &= L\eta d\xi\, dm \\ &= L\, d\xi\, \eta \frac{w}{n} \end{aligned} \tag{13}$$

where $d\xi$ is the differential fraction of coolant material melted. Note that w/n is the total mass flow rate per coolant tube.

It is assumed that all heat flow is normal to the coolant stream. Using this assumption and the assumption of steady-state operation the heat generation rate in the volume cooled by the i^{th} coolant tube Q_i is given by

$$Q_i = \int_{x=0}^{x=l'} dq_i(x) \tag{14}$$

The total heat generation rate Q_R for the reactor is then given by

$$Q_R = \sum_{i=1}^{n} Q_i = \sum_{i=1}^{n} \int_{x=0}^{x=l'} dq_i(x) \tag{15}$$

The volumetric heat source is, in general, a point function S(x,y,z). An effective area A_i can be defined such that the heat generated in the small volume cooled by the length dx of the coolant tube becomes

$$dq_i(x) = \left[\int_{A_i} S(x,y,z)\, dydz \right] dx \tag{16}$$

In the most common reactor designs the coolant tubes are placed parallel to one another and the effective areas can be considered independent of x. Therefore the local average heat source per unit volume $S_i(x)$ is defined by

$$S_i(x) = \frac{1}{A_i} \iint_{A_i} S(x,y,z)\, dydz \qquad (17)$$

From Equations 16 and 17 one then gets

$$dq_i(x) = A_i S_i(x) dx \qquad (18)$$

Combining Expressions 13 and 18

$$d\xi = \frac{nA_i S_i(x)}{L\eta w}\, dx \qquad (19)$$

Integrating 19 over the length of the i^{th} coolant tube gives

$$\int_{\xi(0)}^{\xi(l')} d\xi = \int_0^{l'} \frac{nA_i S_i(x) dx}{L\eta w}$$

$$(\xi(l') - \xi(0) \Big|_{i^{th}\ \text{tube}} = \frac{nA_i}{L\eta w} \int_0^{l'} S_i(x) dx \qquad (20a)$$

$$= \frac{n}{L\eta w} \int_{x=0}^{x=l'} dq_i$$

$$= \frac{n}{L\eta w}\, Q_i \qquad (20b)$$

This shows that the change in fraction of melted coolant material is proportional to the amount of heat generated in the volume cooled by the coolant tube. In order to meet the conditions that the mean temperature of the coolant stream be the same at Points 1 and 2 of Figure 1 and that the latent heat is the mode of heat absorption, Q_i of Equation 20b will always be such that

$$0 \leq \xi(1') - \xi(0) \Big|_{i^{th}\ \mathrm{tube}} \leq 1.00 \qquad (21)$$

While the mean temperature of the coolant stream may be at or near the melting temperature of the coolant material at Points 1 and 2 of Figure 1, the mean temperature of the coolant stream must be higher than the melting temperature while the coolant stream is within the reactor. In addition the temperature of the surface of the coolant tube T_s must be higher than the mean coolant stream temperature T_m. These temperature differences are necessary to allow for the transfer of heat into the coolant stream and from the coolant stream into the coolant particles. In order to simplify the expression for heat transfer from tube to coolant the normal procedure is to assume that an expression of the form

$$dq_i(x) = h(T_s(x) - T_m)\, dA_s \qquad (22)$$

holds at the surface where h is the heat transfer coefficient. The differential area dA_s is the surface area of the coolant tube associated with $dq_i(x)$. In the present case of a uniform circular coolant tube dA_s becomes

$$dA_s = 2\pi\, R_t\, dx \qquad (23)$$

Substituting this result into Equation 22 and integrating over the length of the i^{th} coolant tube one gets

$$Q_i = \int_{x=0}^{x=l'} dq_i(x) = 2\pi\ R_t \int_0^{l'} h(T_s(x) - T_m)dx \tag{24}$$

By combining Equations 22 and 23 and comparing this result with Equation 18 it is seen that

$$A_i S_i(x) = 2\pi\ R_t h(T_s(x) - T_m) \tag{25a}$$

or

$$T_s(x) = \frac{A_i S_i(x)}{2\pi R_t h} + T_m \tag{25b}$$

This equation can be used to determine the surface temperature of the coolant tube if $S_i(x)$ can be determined. The function $S_i(x)$ can be determined by noting the relationship between thermal neutron flux $\phi_{th}(x,y,z)$ and energy generation from Glasstone and Edlund (4,p.80). It is known that it takes approximately 3.1×10^{10} fissions to generate one watt-second of energy. The rate at which fissions occur is the product of thermal neutron flux and the fission cross-section Σ_f. With this information the energy generation in a differential element of volume dVol becomes

$$S(x,y,z)\ dVol = K\ \ \phi(x,y,z)\ dVol \tag{26a}$$

$$= \frac{\Sigma_f}{3.1 \times 10^{10}}\ \phi(x,y,z)dVol \text{ watts} \tag{26b}$$

Therefore

$$Q_R = \int_{Reactor} S_i(x,y,z)dVol \tag{27a}$$

$$Q_R = K \int_{\text{Reactor}} \phi_{th} (x,y,z)dx\ dy\ dz \tag{27b}$$

From Equations 16 and 26

$$dq_i(x) = K \left[\iint_{A_i} \phi_{th}\ (x,y,z)\ dy\ dz\right] dx \tag{28a}$$

If the local average flux is defined as

$$\phi(x) = \frac{1}{A_i} \int_{A_i} \phi_{th}(x,y,z)\ dy\ dz \tag{28b}$$

then $dq_i(x)$ becomes

$$dq_i(x) = KA_i\ \phi(x)\ \ dx \tag{28c}$$

If the constant KA_i is redefined as K_i, Equation 14 becomes

$$Q_i = K_i \int_0^1 \phi(x)dx \tag{29}$$

From Equation 15 the heat removed by the i^{th} coolant tube is just some fraction B_i of the total heat removed from the reactor. Thus

$$Q_i = B_i Q_R \tag{30}$$

Which when combined with Equation 29 gives

$$B_i Q_R = K_i \tag{31}$$

Therefore the fraction of heat removed by the i^{th} coolant tube B_i becomes a weighting function for the i^{th} tube. If it is possible to determine this fraction in some manner when (x) is known, then K_i can be determined. Since from Equations 18 and 29

$$A_i S_i(x) = K_i\ \phi(x) \tag{32}$$

then $T_s(x)$ can be determined from Equation 25_b.

It can be noted that from the assumption of a well mixed coolant stream that $\xi_i(0)$ of Equation 20a is the same for all the coolant tubes and is equal to ξ_1 of Equation 8. However $\xi_i(1)$ will be different for different coolant tubes. The quantity $\xi_i(1)$ is not the actual fraction of the melted coolant material at the exit of the coolant tube. It is instead the fraction of melted coolant material which would exist if the coolant stream were allowed to reach thermal equilibrium. As mentioned above the coolant stream can not be in thermal equilibrium while it is in the reactor. If it were, no heat transfer to the coolant particles would take place. The thermal energy represented by the difference between the actual fraction of melted coolant material and $\xi_i(1)$ is contained in the coolant stream as the heat absorbed by the change in temperature of the materials comprising the coolant stream. If the recombined coolant stream is allowed to reach thermal equilibrium at some later time Equation 20b will represent the heat energy removed by the i^{th} coolant tube. It has already been assumed that the coolant stream is well mixed at the melting temperature of the coolant material t_{melt} at Point 2 in Figure 1. Thus using Equations 15 and 20b it can be seen that

$$Q_R = \sum_{i=1}^{n} \frac{L\eta w}{n} \left[\xi_i(1') - \xi(0) \right] \qquad (33a)$$

or

$$Q_R = L\eta w \left[\sum_{i=1}^{n} \frac{\xi_i(1')}{n} - \xi(0) \right] \qquad (33b)$$

By noting that $\xi(0)$ is equal to ξ_1 and comparing Equation 33b with Equation 8 ξ_2 becomes

$$\xi_2 = \sum_{i=1}^{n} \frac{\xi_i(1')}{n} \qquad (34)$$

Using Equation 20b to solve for $\xi_i(1)$ and substituting into Equation 34, it is seen that

$$\xi_2 = \sum_{i=1}^{n} \frac{Q_i}{L\eta w} + \frac{\xi(0)}{n} \qquad (35)$$

To use the heat absorption power of the latent heat completely, the coolant material would be all solid at Point 1 $\xi(0) = 0$ and all liquid at Point 2 ($\xi_2 = 1.0$). One way for ξ_2 to be unity is for each $\xi_i(1')$ to be unity. For uniform tubes this further implies that the Q_i's are all equal. Thus for the most efficient use of all the latent heat all the Q_i's must be equal. If the Q_i's are not all equal then $\xi_i(1')$ can not all be unity unless the coolant tubes are of non-uniform size or more thermal energy is allowed into some coolant tubes than is necessary to melt the coolant material flowing in those tubes.

Pumping Power

Pumping power as given by Glasstone and Sesonske (5, p. 396) is

$$\text{Pumping Power} = \text{Pressure Drop} \times \text{Volume Flow Rate} \tag{36}$$

If Δ_p stands for pressure drop across the whole system, the symbolic representation of 36 becomes

$$P_p = \Delta p \frac{w}{\rho_c} \tag{37}$$

The pumping power requirement should be as low as possible since pumping power subtracts from the net power of the system. In order to have lower pumping power requirements the volume flow rate term of Equation 36 should be as small as possible. The volume flow rate F is given by

$$F = w/\rho_c \tag{38}$$

Since the density of the coolant under discussion does not vary significantly, the volume flow rate is constant throughout the system. If Equation 8 is solved for w and combined with Equation 38, the volume flow rate becomes

$$F = \frac{Q_R}{\eta \rho_c (\xi_2 - \xi_1) L} \tag{39}$$

The quantity Q_R will normally be fixed by the power generation in the reactor. If the flow rate is to be a minimum, then the terms in the denominator of Equation 39 must be maximized. This implies that $\xi_2 - \xi_1$ be unity which was described earlier as the condition for most efficient use of the latent heat.

If the shell material is neglected, the relationship between the weight fraction of coolant material η, density ρ, and volume V are given by

$$\rho_c = \frac{\text{Total Weight}}{\text{Total Volume}} = \frac{\rho_m V_m + \rho_{Tm} V_{Tm}}{V_c} \tag{40a}$$

$$\eta = \frac{\text{Weight Of Coolant Material}}{\text{Total Weight}} = \frac{\rho_m V_m}{\rho_c V_c} \tag{40b}$$

$$V_c = V_m + V_{Tm} = \text{Constant} \tag{40c}$$

where the subscripts c , m, and Tm designate total coolant stream, coolant material, and transport media respectively. The total volume V_c is constant for a fixed system, and if a choice of materials has been made the density ρ_m is also fixed. Thus from Equation 40

$$\eta \rho_c = \frac{\rho_m V_m}{V_c} \tag{41}$$

which implies that the product $\eta\rho_c$ can be increased by increasing the amount of coolant material. There is, of course, some upper limit to the amount of coolant material that can be added to the stream and still maintain the desired flow conditions.

In order to increase the latent heat L a material change is required. When the material is changed the density ρ_m is also changed. A means of comparison between different materials is possible by noting from Equation 41 and 39 that

$$F = \frac{Q_R V_c}{V_m \rho_m L(\xi_2 - \xi_1)} \qquad (42)$$

Therefore the volume flow rate is inversely proportional to the product $\rho_m L$. Values for this product are given as $\Delta H_m \rho$ in Table 1.

The volume flow rates for the other two thermal energy absorption methods become

$$F_{\Delta T} = \frac{Q_R}{\rho_c c_c \Delta T} \qquad (43a)$$

$$F_V = \frac{Q_R}{\rho_c L_V} \qquad (43b)$$

No ratios of volumes appear in these expressions because the coolant material usually comprises all of the coolant stream for these methods. A comparison of the three values of volume flow rates given by Equations 42, 43a, and 43b shows that for equal values of $\Delta H \rho$ as given by Table 1, the volume flow rate for the latent heat process Equation 42, is higher by the factor V/V_m. When this result is used in conjunction with the equation for pumping power Equation 37, it shows that the pumping power for the latent heat process can be expected to be higher than for the other methods.

THE PARTICLE

The particles described in the last chapter must absorb, store, and release the thermal energy transported by the stream. The whole system will depend on how fast and efficiently these particles can absorb and release the heat energy.

From Figure 1 the thermal changes in a typical particle can be followed as it makes one complete circuit of the coolant loop. The particle starts out at Point 1 at constant temperature with some fraction of the coolant material melted. As the particle moves into the reactor coolant tube, the temperature distribution changes giving the particle a hotter surface temperature. This thermal gradient will cause thermal energy to flow into the coolant material. This in turn causes more of the coolant material to be melted. After the particle passes out of the reactor, a thermal gradient will continue to exist until thermal equilibrium of the coolant stream is established at Point 2. At this point a different fraction of the coolant material is melted, and the temperature is again constant. (Thermal equilibrium need not be established, but it is assumed for discussion purposes.) The particle now enters the heat exchanger where the surface temperature is lowered. Heat is transferred out of the particle. Finally, the particle leaves the heat exchanger and continues to lose its thermal energy to the coolant stream until thermal equilibrium is once again reached at Point 1.

In the discussion of the heat transfer system the assumption of steady-state was taken to imply that the fractions of coolant material melted ξ_1 and ξ_2 at Points 1 and 2 respectively would be constant. While this holds for the total stream, it may not hold for the individual particles. Therefore the fraction of coolant material melted for a particular particle would in general be different at Point 1 at each particle pass through the coolant loop.

In previous discussions it was noted that the rate at which the thermal energy is absorbed by the particles should be close to the rate at which thermal energy is added to the stream. It was pointed out that if this condition did not exist, the excess thermal energy would act to raise the temperature of the stream. Therefore the latent heat would not be used effectively. One measure of the heat absorption rate is the melting time for the coolant material in the particle.

Melting Time

The time necessary for the melting of a fraction of coolant material in one pass through the reactor is directly related to the mass flow rate w. This fraction of coolant material is melted as the particles pass through the reactor to Point 2. Since under the steady-state condition this will be some fixed length, the time of passage through this length must correspond to the melting time for the fraction $\xi_2 - \xi_1$ of the coolant material.

For the purpose of determining melting time, the temperature at the outside surface of the shell is taken to be some function of position in the coolant cycle. It is convenient to choose the point where melting begins as the entrance to the coolant tube x = 0, and the point where melting has essentially stopped as $x = l_m$. Thus the length of coolant channel over which melting takes place is l_m. In the idealized case where the thermal energy is absorbed by the latent heat as fast as it is added to the stream, l_m is the same as the coolant tube length. The velocity of the stream times the cross-sectional area of the coolant tube A_T gives the volumetric flow rate; thus

$$vA_T = \frac{w}{n\rho_c} \qquad (44)$$

The velocity is given by

$$v = \frac{l_m}{t_{me}} \qquad (45)$$

since the melting time t_{me} must be equal to the time of passage through the melting length. Thus combining 44 and 45

$$w = \frac{l_m A_T n}{t_{me}} \qquad (46a)$$

or

$$t_{me} = \frac{l_m A_T n \rho_c}{w} \qquad (46b)$$

The Melting Problem

The particles consist of three distinct parts; shell material, melted coolant material, and solid coolant material. The nature of the melting-solidification process dictates that both melting and solidification start on the outside edge of the coolant material and proceed toward the center. This combined with the fact that the fraction of coolant material melted for any particular particle changes on each complete circuit of the coolant loop, leads to the conclusion that the particle may consist of alternate layers of melted and solidified coolant material. During melting the outside layer will always be melted coolant material. To circumvent the problem of handling several layers, all of the coolant material will be assumed to melt during passage through the reactor and resolidify during passage through the heat exchanger. This condition has already been pointed out as one of the most efficient use of the latent heat for heat transfer.

It will be further assumed that the properties of the materials are independent of temperature in the range of interest. For the j^{th} region one then has

$$\alpha_j \nabla^2 T = \frac{\partial T}{\partial t} \qquad (47)$$

where j is either sh, L, or c corresponding to shell, liquid, or solid respectively. This is the time dependent heat conduction equation. The thermal diffusivity α_j of the region is given by

$$\alpha_j = \frac{K_j}{\rho_j c_j} \qquad (48)$$

where the material density ρ_j the specific heat C_j and the thermal conductivity K_j are all properties of the material for the j^th region.

Applying a set of boundary conditions between the liquid and solid material, one then obtains a nonlinear problem. This is caused by the fact that the boundary between the liquid and solid region is moving and changing with time. Several authors have presented solutions to this type of problem. Among them are Altman et al. (1), Baxter (2), Goodman (6), Goodman and Shea (7), Ingersoll and Zobel (12), Landau (13), Murray and Landis (16), O'Brien et al. (17), Ross (19), and Stolz (20). Without exception a one dimensional geometry is used. Because of its adaptability to spherical geometry a method illustrated by Murray and Landis (16) is used in the following development.

In order to reduce the problem to one dimension, one must assume that the particles are spherical with a known time dependent uniform temperature distribution on the outside surface of the shell. If it is further assumed that the unmelted part of the coolant material stays centered in the shell, one obtains a symmetrical problem. With these assumptions the heat conduction Equation 47 becomes

$$\frac{\partial T}{\partial t} = \frac{a_j}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) \qquad (49a)$$

which can be resolved into

$$\frac{\partial T}{\partial t} = \frac{2a_j}{r}\frac{\partial T}{\partial r} + a_j\frac{\partial^2 T}{\partial r^2} \qquad (49b)$$

The boundary condition at the outside surface of the shell is

$$T(r = R_2) = T_{boun}(t) \qquad (50)$$

where the symbols are defined in Figure 3. The two boundary conditions at the interface between the shell and the liquid are that the temperature and heat flux be continuous. Thus

$$T_{sh}(r = R_1) = T_L(v = R_1) \qquad (51)$$

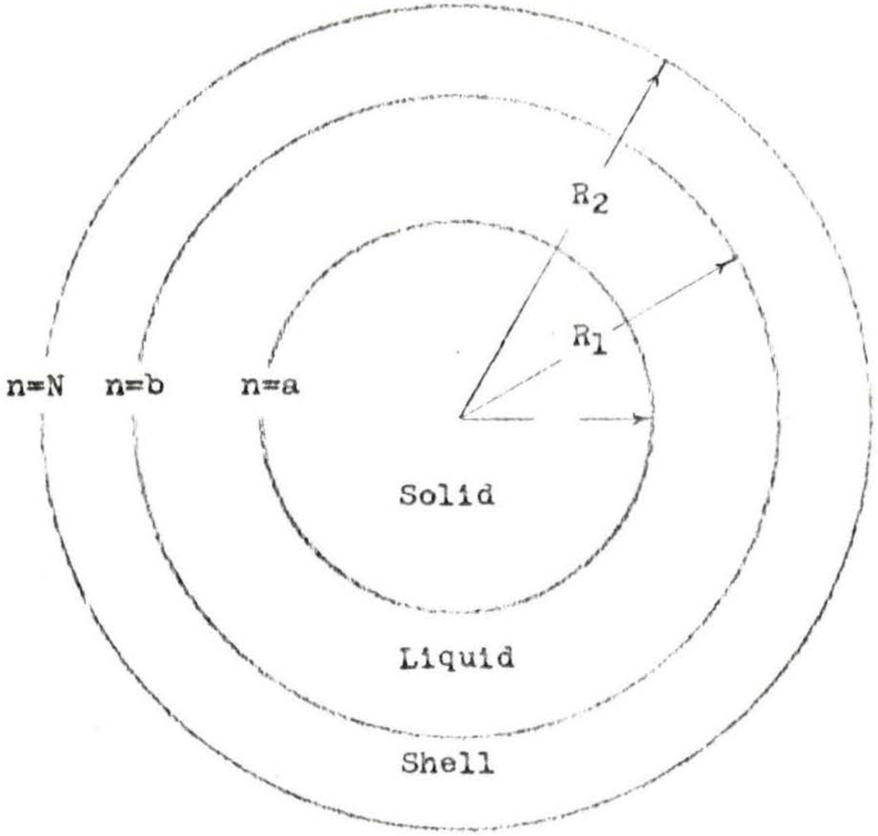
and

$$K_{sh}\frac{\partial T_{sh}}{\partial r} = K_L\frac{\partial T_L}{\partial r}, \quad r = R_1 \qquad (52)$$

For one boundary condition at the liquid-solid interface one must take an energy balance. The thermal energy flowing into the interface is equal to the thermal energy absorbed by the melting process plus the thermal energy which flows away from the interface. Mathematically this becomes

$$K_L\frac{T_L}{r} = \rho_s L\frac{\partial \epsilon}{\partial t} + K_s\frac{\partial T_s}{\partial r}, \; r = \epsilon \qquad (53)$$

The temperature at the liquid-solid interface must be at the melting temperature of the coolant material,



Shell, $R_1 \leq r \leq R_2$, $b \leq n \leq N$

Liquid, $\epsilon \leq r \leq R_1$, $a \leq n \leq b$

Solid, $0 \leq r \leq \epsilon$, $0 \leq n \leq a$

Figure 3. Spherical coolant particle.

$$T_L(r = \epsilon) = T_s(r = \epsilon) = T_{melt} \qquad (54)$$

The final boundary condition occurs at the center. From the symmetry of the problem there can be no thermal energy flow at the center, thus

$$K_s \frac{\partial T_s}{\partial r} = 0 \quad , r = 0 \qquad (55)$$

The methods shown by Murray and Landis use numerical techniques to adapt this problem to either the analog or digital computers. They term the method adapted here as the "variable space network." A three point numerical approximation is used for the spatial partial differentials involved. Thus

$$\frac{\partial T_n}{\partial r} = \frac{T_{n+1} - T_{n-1}}{2\Delta r|_j} \qquad (56)$$

and

$$\frac{\partial^2 T_n}{\partial r^2} = \frac{T_{n-1} - 2T_n + T_{n+1}}{(\Delta r|_j)^2} \qquad (57)$$

Here the subscript n refers to the n^{th} spherical surface as defined in Figure 3. The radial differences $\Delta r|_j$ for each region are given by

shell region:

$$\Delta r|_{sh} = \frac{R_2 - R_1}{n - b} \qquad (58)$$

liquid region:

$$\Delta r\big|_L = \frac{R_1 - \epsilon}{b - a} \tag{59}$$

solid region:

$$\Delta r\big|_s = \frac{\epsilon}{a} \tag{60}$$

Using these definitions for the radial differences the radius to the n^{th} surface, r_n is given by

shell region:

$$r_n = (n-b)\Delta r\big|_{sh} + R_1 \tag{61}$$

liquid region:

$$r_n = (n-a)\Delta r\big|_L + \tag{62}$$

solid region:

$$r_n = n\Delta r\big|_s \tag{63}$$

The finite difference relations will be different for each region. Considering the shell region first Equations 49b, 56, and 57 combine to give

$$\frac{\partial T_n}{\partial t} = \frac{\alpha_{sh}(T_{n+1}-T_{n-1})}{r_n \ \Delta r\big|_{sh}} + \alpha_{sh}\frac{(T_{n-1}-2T_n+T_{n+1})}{(\Delta r\big|_{sh})^2} \tag{64}$$

Next considering the liquid region the rate of travel of the surface is related to the interface velocity by

$$\frac{dr_n}{dt} \Big/ R_1 - r_n = \frac{d\epsilon}{dt} \Big/ R_1 - \epsilon \tag{65}$$

from the definition of the total derivative

$$dT = \frac{\partial T}{\partial r}\, dr + \frac{\partial T}{\partial t}\, dt \tag{66a}$$

one has

$$\frac{dT_n}{dt} = \frac{\partial T_n}{\partial r} \frac{dr_n}{dt} + \frac{\partial T_n}{\partial t} \tag{66b}$$

Substituting Equation 49b into 66b one has

$$\frac{dT_n}{dt} = \frac{\partial T_n}{\partial r} \frac{dr_n}{dt} + \frac{2\alpha_L}{r_n} \frac{\partial T_n}{\partial r} + \alpha_L \frac{\partial^2 T_n}{\partial r^2} \tag{67}$$

Here the subscript n in the derivative terms denotes that the derivative is evaluated at the nth point. Now Equations 67, 65, 57 and 56 can be combined to give

$$\begin{aligned} \frac{dT_n}{dt} = {} & \frac{(T_{n+1} - T_{n-1})}{2\Delta r|_L} \cdot \frac{(R_1 - r_n)}{(R_1 - \epsilon)} \cdot \frac{d\epsilon}{dt} \\ & + \frac{2\alpha_L(T_{n+1} - T_{n-1})}{r_n\,(2\Delta r|_L)} \\ & + \frac{L(T_{n-1} - 2T_n + T_{n+1})}{(\Delta r|_L)^2}, \quad a < n < b \end{aligned} \tag{68}$$

For the solid region corresponding results can be shown, that is

$$\frac{dr}{dt} / r_n = \frac{d\epsilon}{dt} / \epsilon, \quad 0 < n < a \tag{69}$$

and

$$\frac{dT_n}{dt} = \frac{\partial T_n}{\partial r} \frac{dr_n}{dt} + \frac{2\alpha_s}{r_n} \frac{\partial T_n}{\partial r} + \alpha_s \frac{\partial^2 T_n}{\partial r^2} \tag{70}$$

Combining Equations 69, 57, and 56 with Equation 70 one has the results

$$\frac{dT_n}{dt} = \frac{(T_{n+1} \; - T_{n-1})}{2\Delta r|_s} \cdot \frac{r_n}{\epsilon} \cdot \frac{d\epsilon}{dt}$$

$$+ \frac{2\alpha_s \, (T_{n+1} \; - T_{n-1})}{r_n \, 2(\Delta r|_s)}$$

$$+ \frac{\alpha_s \, (T_{n-1} \; - 2T_n + T_{n+1})}{(\Delta r|_s)^2}, 0 < n < a \qquad (71)$$

The boundary conditions can also be put into finite difference form. However, the value of the derivatives at the boundary points must be defined differently since the material properties change at the boundaries. To obtain an expression for these derivatives the following Taylor series expansions were used

$$T_{n \pm 1} \cong T_n \pm (\Delta r|_{n\pm}) \frac{\partial T_n}{\partial r} + \frac{(\Delta r|_{n\pm})^2}{2} \frac{\partial^2 T_n}{\partial r^2} \qquad (72)$$

and

$$T_{n \pm 2} \cong T_n \pm (2\Delta r|_{n\pm}) \frac{\partial T_n}{\partial r} + \frac{(2\Delta r|_{n\pm})^2}{2} \frac{\partial^2 T_n}{\partial r^2} \qquad (73)$$

By eliminating the second partial derivative terms from Equations 72 and 73 a finite difference approximation for the temperature gradient on one side of the boundary point n becomes

$$\left.\frac{\partial T_n}{\partial r}\right|_{\text{boundary}} = \pm \; \frac{4T_{\pm} - T_{n \pm 2} \; - \; 3T_n}{2(\Delta r|_{\pm})} \qquad (74)$$

where the + sign refers to the side away from the center of the sphere and the - sign refers to the side nearest the center. The boundary condition of Equation 50 becomes

$$T_n = T_{boundary} \tag{75}$$

The boundary condition at the shell-liquid interface given by Equation 51 becomes simply

$$T_{sh}(r = R_1) = T_L(r = R_1) = T_b \tag{76}$$

Using 74 the continuity of heat flux at the shell-liquid interface from Equation 52 becomes

$$K_{sh}\,\frac{(4T_{b+1} \;\; -T_{b+2} \;\; -3T_b)}{2(\Delta r|_{sh})} = -\,K_L\,\frac{(4T_{b-1} \;\; -T_{b-2} \;\; -3T_b)}{2(\Delta r|_L)} \tag{77}$$

The boundary condition at the liquid-solid interface from Equation 53 combined with Equation 74 gives an expression for the rate of movement of the interface

$$\frac{d\epsilon}{dt} = \frac{1}{\rho_{sL}}\left[K_s\frac{(4T_{a-1} \;\; -T_{a-2} \;\; -3T_a)}{2(\Delta r|_s)} + K_L\frac{(4T_{a+1} \;\; -T_{a+2} \;\; -3T_a)}{2(\Delta r|_L)}\right] \tag{78}$$

Continuity of temperature at the liquid-solid boundary 54 offers no difficulty since

$$T_L(r = \epsilon) = T_s(r = \epsilon) = T_{melt} = T_a \tag{79}$$

The final boundary condition at the center given by Equation 55 combined with Equation 74 becomes

$$K_S\frac{(4T_1 - T_2 - 3T_0)}{2(\Delta r|_S)} = 0 \qquad (80)$$

Initial conditions are necessary to use these equations. It has been assumed in formulating these equations that there were three regions. This is not strictly true since at $t = 0$ i.e., upon entry into the coolant tube, there are only two regions, shell and solid coolant material. Since $\Delta r|_L$ and $\Delta r|_S$ appear in the denominators of several of the expressions they cannot be assigned values arbitrarily close to zero. Therefore, some initial liquid thickness must be assumed, and in the computations the solid material does not completely melt. In addition to this initial liquid thickness, an initial temperature distribution is necessary. In order to find these initial conditions it is possible to calculate the temperature distribution in a composite sphere without melting. An analytical form is shown by Carlslaw and Jeager (3, p. 351). The analytical form is difficult to evaluate since it requires evaluating the roots of a trigonometric relationship. Each root is then used to evaluate one term of an infinite series. On the other hand, computer formulations are relatively easy to construct and their accuracy can be made at least as good as the present formulation of the melting problem. The computer formulations are also more versatile in that almost completely arbitrary boundary and initial conditions may be used. Also temperature varying material properties can be included. It is then possible to combine this type of computer formulation with a formulation of the melting problem to give the temperature distribution at anytime after start-up of the coolant flow cycle. Because of the amount of computer time required, the initial conditions will be assumed for this presentation.

Most of the comments above apply equally well to both digital and analog devices. The finite difference equations developed to this point are in a form which is adaptable to analog computers. The results of an analog computer would be desirable over those of a digital computer because of the continuous time variable inherent in analog computers. However, because of the non-linear nature of the equation, a large amount of expensive equipment is necessary to program these equations for enough points to get a reasonable solution. The equipment was not available, therefore this method of solution was not attempted.

The difference equations given above can be programmed for the digital computer by putting the time dependence as well as the spacial dependence into finite difference form. This can easily be done with the following approximations

$$\frac{dT_n}{dt} = \frac{T_{n,m+1} - T_{n,m}}{\Delta t_{m+1}} \tag{81}$$

and

$$\frac{d\epsilon}{dt} = \frac{\epsilon_{m+1}}{\Delta t_{m+1}} = \frac{\epsilon_{m+1} - \epsilon_m}{\Delta t_{m+1}} \qquad (82)$$

where the subscript m is used to denote the m^{th} time interval. The time interval Δt_m is given by

$$\Delta t_m = t_m - t_{m-1} \, , \quad m = 1, 2, \ldots \qquad (83)$$

so that the total time is given by

$$t_m = \sum_{i=1}^{m} \Delta t_i \qquad (84)$$

In addition let Δr_n and r_n be defined by Equations 58, 59, 60, 61, 62, and 63 where ϵ is replaced by ϵ_m. The results of these modifications as applied to Equations 78, 71, 68 and 64 are given in Table 2.

If the temperature distribution at time $t = t_m$ is known, there are N + 1 unknown temperatures for $t = t_{m+1}$. With the temperature at the boundary given at time $t = t_{m+1}$ there are N-b-2 equations from 85, b-a-2 equations from 86, a-1 equation from 87, and one equation each from 89, 88, 80, and 79, and 75. Adding these equations up there are N + 1 equation in the N + 1 unknown temperatures, so that it should be possible to solve for all of the unknown temperatures.

Most finite difference formulations require that the increments used satisfy some relationship in order for the formulation to be stable. This relationship will depend on

Table 2. Finite difference equations.

Equation		Range	Number of Equations	Reference Equations
Shell $T_{n,m+1} = T_{n,m} + \frac{\Delta t_{m+1}\,\alpha_{sh}(T_{n+1}-T_{n-1})}{r_n \Delta r\vert_{sh}} + \frac{\Delta t_{m+1}\,\alpha_{sh}(T_{n-1}-2T_n + T_{n+1})}{(\Delta r\vert_{sh})^2}$	(85)	$N-1 \geq n \geq b+1$	N-b-2	(64),(49b)
Liquid $T_{n,m+1} = T_{n,m} + \frac{(T_{n+1}-T_{n-1})\ (R_1-r_n)\Delta\epsilon_{m+1}}{2\Delta r\vert_L (R_1-\epsilon_m)} + \Delta t_{m+1} \frac{\alpha_L (T_{n-1}-T_{n-1})}{r_n \Delta r\vert_L} + \frac{\Delta t_{m+1}\,\alpha_L(T_{n-1}-2T_n + T_{n+1})}{(\Delta r\vert_L)^2}$	(86)	$b-1 \geq n \geq a+1$	b-a-2	(68),(67), (65),(49b)
Solid $T_{n,m+1} = T_{n,m} + \frac{(T_{n+1}-T_{n-1})\, r_n \;\; \epsilon_{m+1}}{2\Delta r\vert_s \;\; \epsilon_m} + \frac{\Delta t_{m+1}\,\alpha_s(T_{n+1}-T_{n-1})}{r_n \Delta r\vert_s} + \frac{\Delta t_{m+1}\,\alpha_s(T_{n-1}-2T_n + T_{n+1})}{(\Delta r\vert_s)^2}$	(87)	$a-1 \geq n \geq 1$	a-1	(71),(70), (69)

Table 2. (Continued)

Equation		Range	Number of Equation	Reference Equations
Boundary Conditions				
Liquid-Solid Interface	(88)	n=a	1	(78),(53)
$\epsilon_{m+1} = \epsilon_{m+1} - \epsilon_m = \frac{\Delta t_{m+1}}{\rho_s L}\left[\frac{K_s}{2(\Delta r\|_s)}\left(4T_{a-1,m} - T_{a-2,m} - 3T_{a,m}\right) + \frac{K_L}{2\Delta r\|_L}\left(4T_{a+1,m} - T_{a+2,m} - 3T_{a,m}\right)\right]$				
$T_{a,m} = T_{melt} = \text{constant}$	(79)	n=a	1	(79),(54)
Shell-Liquid Interface	(89)	n=b	1	(77),(52)
$T_{b,m} = \frac{\Delta r\|_{sh}\Delta r\|_L}{3K_{sh}\Delta r\|_L + 3K_L\Delta r\|_{sh}}\left[\frac{K_{sh}}{\Delta r\|_{sh}}\left(4T_{b+1,m} - T_{b+2,m}\right) + \frac{K_L}{\Delta r\|_L}\left(4T_{b-1,m} - T_{b-2,m}\right)\right]$				
Outside Boundary	(75)	n=n	1	(75),(50)
$T_{n,m} = T_{boundary,m}$				
Center Condition	(80)	n=0	1	(80),(55)
$T_{o,m+1} = \frac{1}{3}\left(4T_{1,m+1} - T_{2,m+1}\right)$				

the parameters of the problem and particularly on the choice for the starting position. It turns out that the time increments required by stability are very small for this particular problem. Since these increments are small, a very large number of points must be calculated to obtain the desired solution.

The results as given by Murray and Landis (16) for infinite slab geometry have been compared to the experimental results by Thomas and Westwater. The calculated interface velocity was less than half the experimental velocity. In addition, the experimental velocity peaked in a manner which was not predicted by the numerical solutions. These comparisons would indicate that a high degree of caution is necessary in using the results of these calculations.

With the initial and boundary conditions given, the time that it takes to melt the solid coolant material from radius ϵ_0 to radius ϵ_M can be found. Both ϵ_0 and ϵ_M must be chosen to keep the program stable. The choice of some ϵ_0 as an initial condition is the equivalent to choosing as the starting position some finite distance l_0 from the coolant tube entrance. If $dq_i(x)$ is known, l_0 may be found by integrating Equation 13.

$$\int_{x=0}^{x=l_0} dq_i(x) = \frac{w \eta L(R_1^3 - \epsilon_0^3)}{n \quad R_1^3} \qquad (90)$$

(Note that the fraction of coolant melted is the ratio of the volume of melted coolant material to the total volume of coolant material.) The radius of the solid ϵ_M can be taken as the radius at $x=1'$. In earlier discussions it was suggested that the amount of unmelted material in the coolant stream as the stream leaves the tube be just sufficient to bring the coolant stream into thermal equilibrium at the melting temperature of the coolant material with all the coolant material melted. Here it will be considered sufficient to use

$$\epsilon_M = R_1 - \epsilon_0 \tag{91}$$

If the volume of coolant material not included in the calculations is small compared with the total volume of coolant material, that is if

$$\frac{4}{3}\pi(R_1^3 - \epsilon_0^3) + \frac{4}{3}\pi\epsilon_M^3 \ll \frac{4}{3}\pi R_1^3 \tag{92a}$$

or

$$\epsilon_m^3 - \epsilon_0^3 \ll 0 \tag{92b}$$

then

$$t_{me} = t_m \tag{93}$$

However, if the condition 92 is not met some form of correction must be applied to the calculated melting time t_m. In one possible correction one can assume that the average melting rate of the unaccounted for coolant material is the same as the average melting rate for the calculated result.

$$t_{me} = t_m \left[1 + \frac{\frac{4}{3}\pi(R_1^3 - \epsilon_0^3) + \frac{4}{3}\pi\,\epsilon_m^3}{\frac{4}{3}\pi(\epsilon_0^3 - \epsilon_m^3)} \right] \tag{94a}$$

or

$$t_{me} = t_m \left(\frac{R_1^3}{\epsilon_0^3 - \epsilon_m^3} \right) \tag{94b}$$

The melting time from either Equation 93 or 94 can then be used to determine mass flow rate from Equation 46a.

Particle Size

The particle size has a direct influence on how fast the coolant material can be melted. The size is also an important factor in determining the properties of the coolant stream. It would seem that the melting time should decrease as the size of the particle is decreased. In addition, it would seem reasonable to expect better suspension properties with smaller particles. However, as the particles get smaller, the fraction of shell material may have to be increased, or fabrication may be more difficult. In either case, some method of choosing an optimum size should be used.

Stress in the Shell

For the purposes of discussion, the shell stresses will be classified into three types; mechanical, thermal, and thermal expansive. The mechanically induced stresses are those stresses produced by collisions with other particles and with the walls of the flow channel as well as other stresses produced by the method of transport. These stresses are difficult to evaluate. However, some forethought in design of the coolant system could minimize the effect of these stresses. For example if it could be determined that flow through the pump caused the greatest mechanical stresses than the pumping operation could be placed in a portion of the flow where all the coolant material is solid. In this way, the shell would get the added strength of the solid center. This might not always be the case. Depending on the choice of materials and the shell construction the melted coolant material could be under high pressure. In this case, the high pressure may give added strength to the shell for certain types of stresses.

Thermally induced stresses are considered here to be those stresses caused by thermal gradients. While these may be important in some cases, during normal operation the particles should be small, and the temperature differences should also be small. With these two factors working together, the thermal stresses should be negligible.

The stresses caused by the difference in thermal expansion of the shell and coolant material could be significant. In addition, most materials change density during phase change. This effect could add considerably to the stresses in the shell. Whenever possible through design and fabrication, these stresses should be made to counteract the mechanically induced stresses. There are three cases for stresses caused by differences in thermal expansion corresponding to whether the thermal expansion of the shell is greater than, equal to, or less than the thermal expansion in the coolant material. In the case where the two are equal no stresses are induced. In the other two cases high stresses can be developed by changes in temperature because an interference fit will be caused. It might be possible to avoid these stresses through a choice of fabrication process and temperature, but more likely the shell will have to be designed to be subjected to these stresses without fracture or damage.

An analysis of these shell stresses can be carried out for the relatively simple case of a spherical shell with symmetry. Let p_i be the pressure on the inside and p_o be the pressure on the outside.

Thin Walled Assumption

For the case where the thickness of the shell is small compared to the radius the following method is presented in Higdon et al. (10). Referring to Figure 4 it is noted that the force P exerted on the hemisphere by the pressure is

$$P = p\pi R_1^2 \qquad (95)$$

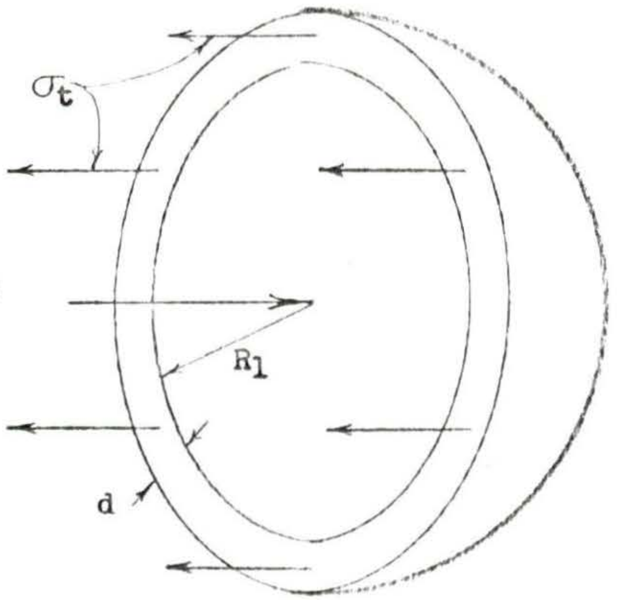


Figure 4. Thin-walled hemispherical pressure vessel.

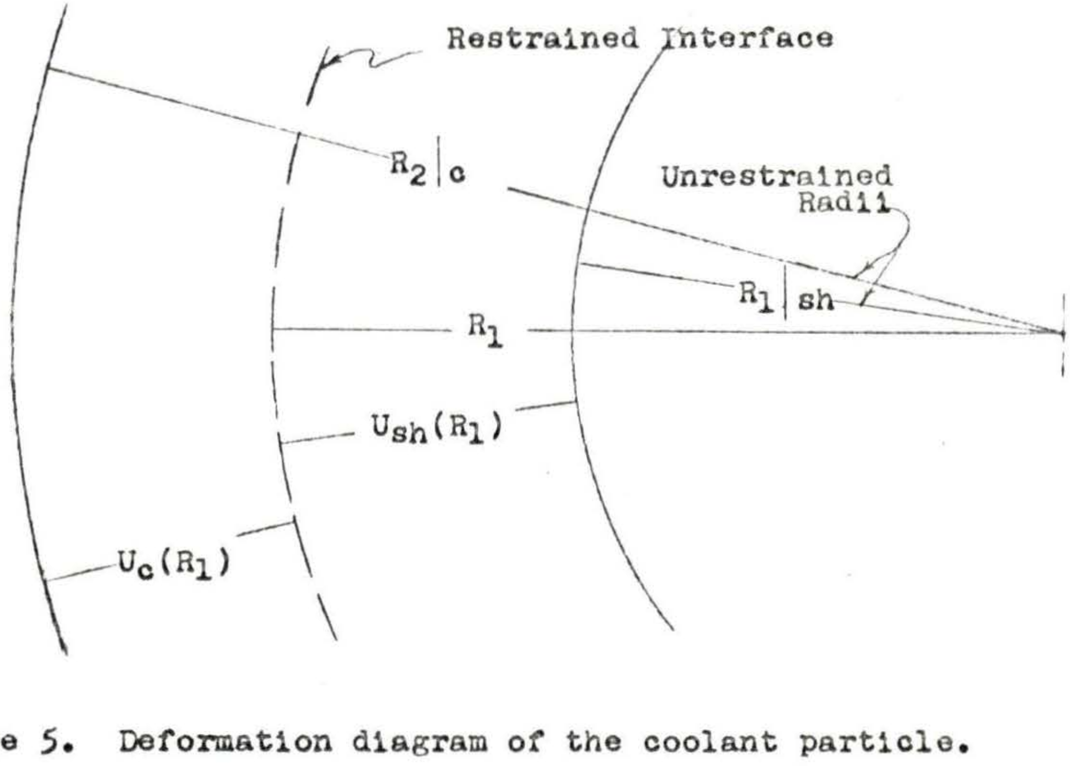


Figure 5. Deformation diagram of the coolant particle.

where the pressure p is taken approximately as

$$p = p_i - p_0 \qquad (96)$$

Since the shell must be in static equilibrium, the force developed in the shell must just equal the pressure induced force, thus

$$2\pi R_1 \sigma d = p\, \pi R_1^2 \qquad (97)$$

The stress in the shell then becomes

$$\sigma = \frac{R_1}{d} \qquad (98)$$

The normal criterion for the thin walled assumption is that

$$\frac{d}{R_1} < 0.1 \qquad (99)$$

If the thin wall assumption does not hold, the more complex analysis of a thick-walled sphere must be used. The stress analysis for thick-walled spheres is given in Appendix B. The results of this analysis are

$$\sigma_t = \frac{p_i R_1^3 - p_0 R_2^3}{R_2^3 - R_1^3} + \frac{R_1^3 R_2^3 (p_i - p_0)}{2R^3 (R_2^3 - R_1^3)} \qquad (127)$$

$$\sigma_r = \frac{p_i R_1^3 - p_0 R_2^3}{R_2^3 - R_1^3} - \frac{R_1^3 R_2^3 (p_i - p_0)}{r^3 (R_2^3 - R_1^3)} \qquad (128)$$

where R_1 and R_2 are the inside and outside radius respectively and σ_t and σ_r are the transverse and radial components of stress. The stress in the coolant material is also given in Appendix B, as

$$\sigma_t\big|_c = - p_0\big|_c \tag{100}$$

$$\sigma_r\big|_c = - p_0\big|_c \tag{101}$$

where the subscript c denotes coolant material.

The radial stress at the inside surface of the shell should equal the radial stress on the outside surface of the coolant material. Thus

$$\sigma_r\big|_{sh} = - p_o\big|_c = p_i \tag{102}$$

Both the shell and the center are changed $U_{sh}(R_1)$ and $U_c(R_1)$ respectively from the unrestrained positions $R_1\big|_{sh}$ and $R_2\big|_c$. From Appendix B Equations 126 and 121 these changes are

$$U_{sh}(R_1) = \frac{R_2^3 R_1}{2\,E\,(R_2^3 - R_1^3)} \quad (p_0\big|_{sh} - p_0\big|_c)\ (1 + M)$$

$$+ 2(p_0\big|_c R_1^3 - p_0\big|_{sh} R_2^3)\ (2M - 1) \tag{103}$$

and

$$U_c(R_1) = - p_0\big|_c \frac{(2M - 1)}{E} R_1 \tag{104}$$

From Figure 5, the sum of the two changes in radii must be equal to the difference between the two unrestrained radii, or

$$U_{sh}(R_1) + U_c(R_1) = R_2\big|_c - R_1\big|_{sh} \tag{105}$$

The radii $R_1\big|_{sh}$ and $R_2\big|_c$ can be found from the linear thermal expansion coefficient k_j and the following expression

$$R = R_0 + k_j \triangle T \qquad (106)$$

where R_0 is the reference radius and $\triangle T$ is the change in temperature. Therefore with all the physical properties known it would be possible to calculate the stress at any point in the shell.

SUMMARY

Suggested Further Study

Four main areas of possible further investigations have been opened in connection with the analysis given in the text. These areas can be listed briefly as; fluid flow, materials, economics, and analysis of the melting problem. In most of these areas it is primarily a matter of reviewing what has already been done to see what might be applicable to using the latent heat of fusion. A further investigation into fluid flow should be attempted in order to find the effect of size, shape, and mass of the particles on such quantities as pressure drop, velocity, corrosion, erosion, and heat transfer coefficients of the fluid. A study of materials should be directed toward choosing materials with the desired properties needed for this method.

Economics will be the final judge of any proposed system. For this reason, any economic analysis would be aimed at comparing a proposed system with existing systems or other possible systems.

The melting problem has been investigated in the past as seen by the references given in the text. A different analytical approach may be helpful. Extensions to more than one dimension is also desirable. Other efforts could be directed toward convective boundary conditions between the solid and liquid phases as well as the effects of changes in material density upon phase change. It might be possible to investigate the melting problem from a microscopic point of view rather than the macroscopic approaches taken before. Perhaps use could be made of the experimentally verified fact (Thomas and Westwater (21)) that the interface is composed of irregular sized crystals with varying rates of growth.

Summary of the Analysis

The analysis was directed mainly at the heat transfer and melting problems involved in using the latent heat of fusion. Other factors besides those of heat transfer may make it difficult to employ the latent heat. but strictly from a heat transfer point of view it is possible and under certain conditions even desirable to make use of the latent heat for heat transfer. The primary advantage of using the latent heat lies in reducing the temperature drop required in the heat source.

APPENDIX A

A computer code was programmed for the IBM 360/50 computer to solve for melting time and temperature distributions using the equations in Table 2. The programming language used was FORTRAN IV with basic programming support. The logic for the program is shown by the flow diagram in Figure 6. Table 3 lists the values used for the material properties. The value of density listed for bismuth is a compromise value since bismuth actually contracts upon melting. The other values in Table 3 were selected as representative values in the temperature range of interest.

Table 4 lists the values of the parameters used in the program. The initial conditions and resulting temperature distributions for the two runs are shown in Figure 7A and 7B. The linear initial temperature distribution was chosen for simplicity. The initial melted thicknesses were chosen in conjunction with the time increments by trial and error using short runs of the program. The values for these parameters were choosen to keep the computer time required reasonable. The boundary conditions were chosen large for the same reason. Even with these choices Run A did not go to completion but was terminated by the operator after twenty minutes. However Run B ran to completion with a total run time of approximately 9.5 minutes.

The effect of round off error could be seen in the last several temperature distributions of Run A. The value of the temperature for the radius next to the solid-liquid interface increased in an unexpected manner until it was larger than the temperature at the next largest radius. This is a physically impossible situation from the nature of heat conduction. This situation is demonstrated by the circled point in Figure 7A. This point was not included in the curve at t = 0.2587 because it was considered to be an erroneous point due to the round off errors.

Equation 82 was used to calculate the interface velocity given in Figures 8A and 8B while the melting rates of the coolant material in Figures 9A and 9B were calculated using

$$\text{melting rate} = 4\pi \epsilon_m^2 \frac{\Delta \epsilon_m}{\Delta t} \rho_s \tag{107}$$

Using Equation 94b to calculate melting time one gets

$$\text{Run A:} \quad t_{me} = \frac{0.2587\ (1)^3}{(0.8)^3 - (0.3560)^3} \tag{108a}$$

$$= 0.554 \text{ sec}$$

$$\text{Run B:} \quad t_{me} = \frac{0.01605\ (0.1)^3}{(0.08)^3 - (0.02)^3} \tag{108b}$$

$$= 0.0318 \text{ sec}$$

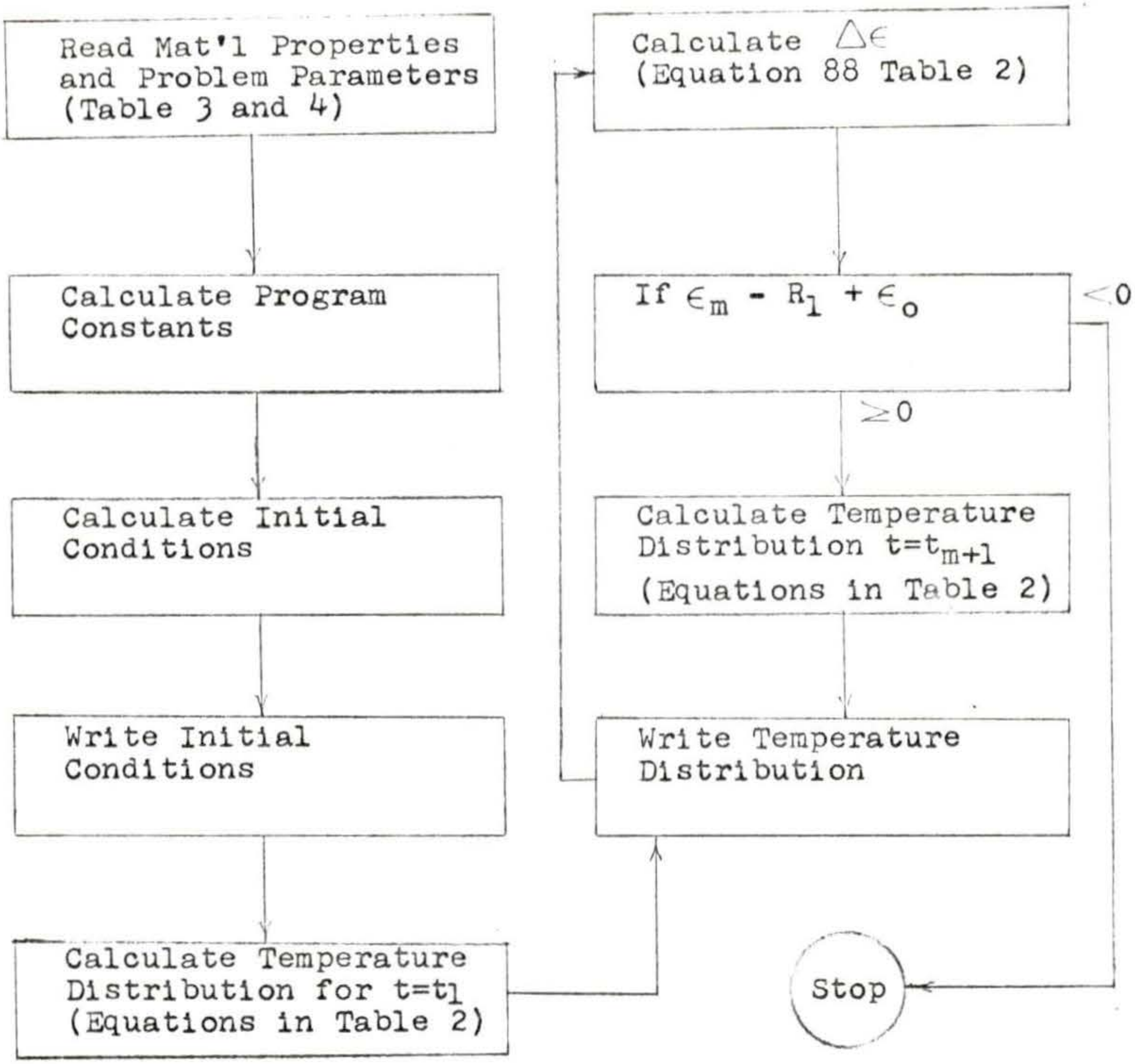


Figure 6. Flow diagram for melting problem problem.

Table 3. Material properties

Material Property	Shell Steel	Liquid Bismuth	Center Bismuth	Dimensions
Specific Heat, C_p	0.11 (3)*	0.03636 (1)	0.03244(1)	cal/g °K
Density,	7.86 (3)	9.75 (4)	9.75(4)	g/cm^3
Thermal Conductivity, K	0.12 (3)	0.03636 (2)	0.41(2)	$\frac{cal}{cm\text{-}sec^{o}K}$
Latent Heat, L			12.44(1)	cal/g
Melting Temperature, T_{melt}			544.5(1)	°K

*Numbers in parenthesis indicate references used for the property values.

Table 4. Program parameters

Parameter	Run A	Run B
Total Number of divisions, N	27	27
Divisions to Shell, b	18	18
Divisions to Liquid, a	9	9
Boundary Temperature, $T_{boundary}$	600°K	560°K
Radius to Shell, R_1	1.0cm	0.1cm
Total Radius, R_2	1.1cm	0.11cm
Time Increment, $\triangle t$	10^{-4}sec	10^{-5}sec
Initial Interface Position, ϵ_o	0.8cm	0.08cm

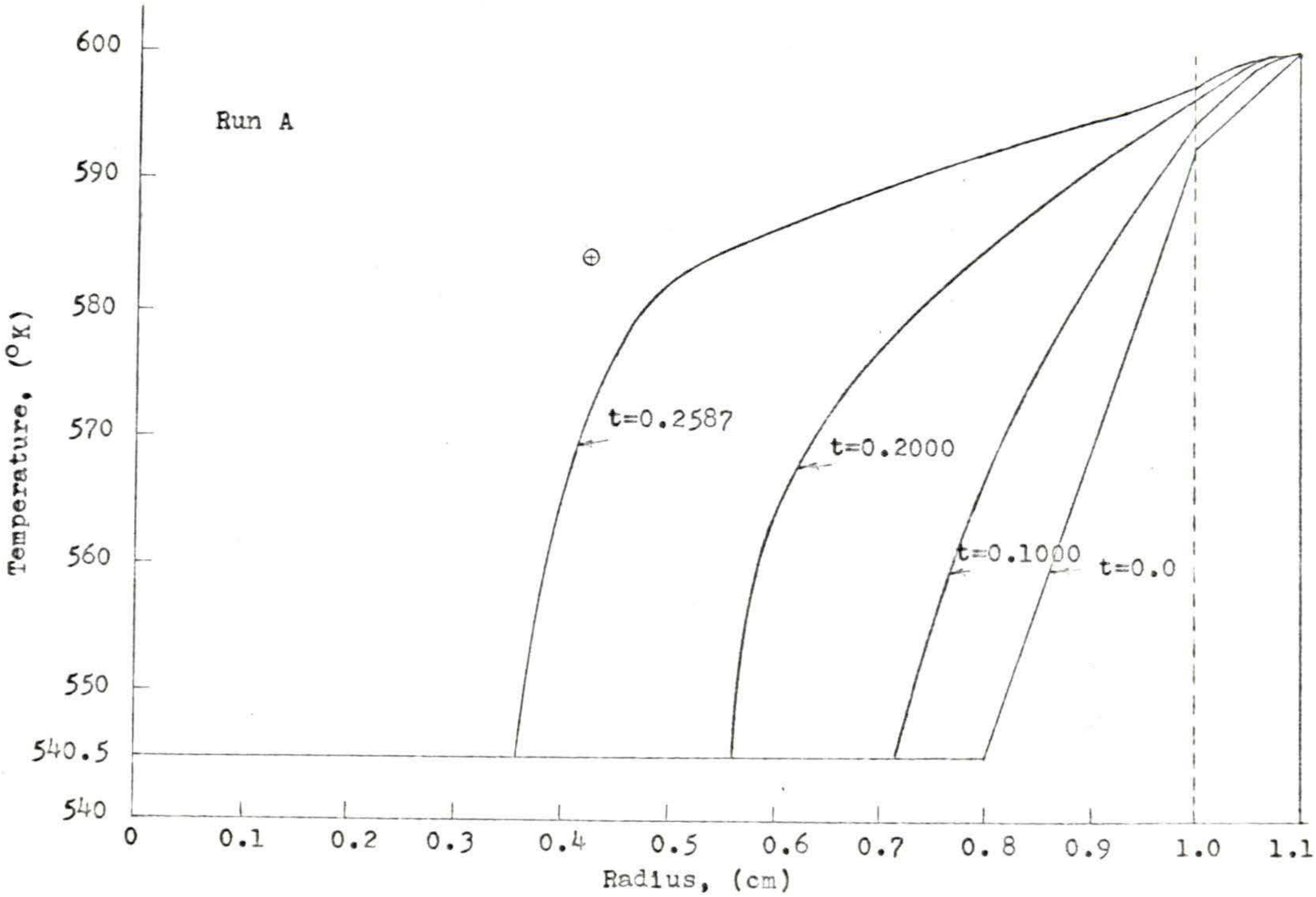


Figure 7A. Temperature distributions.

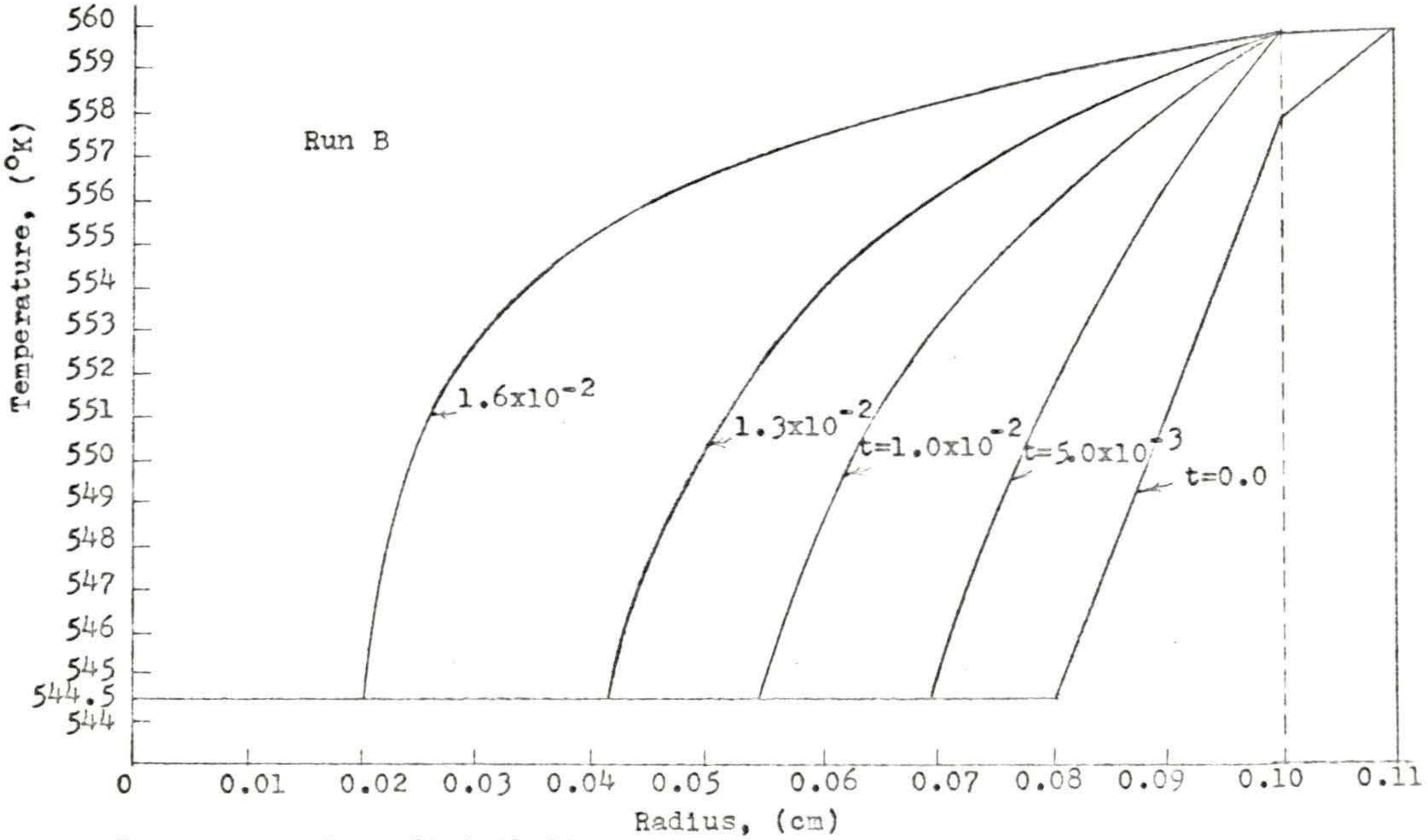


Figure 7B. Temperature distributions.

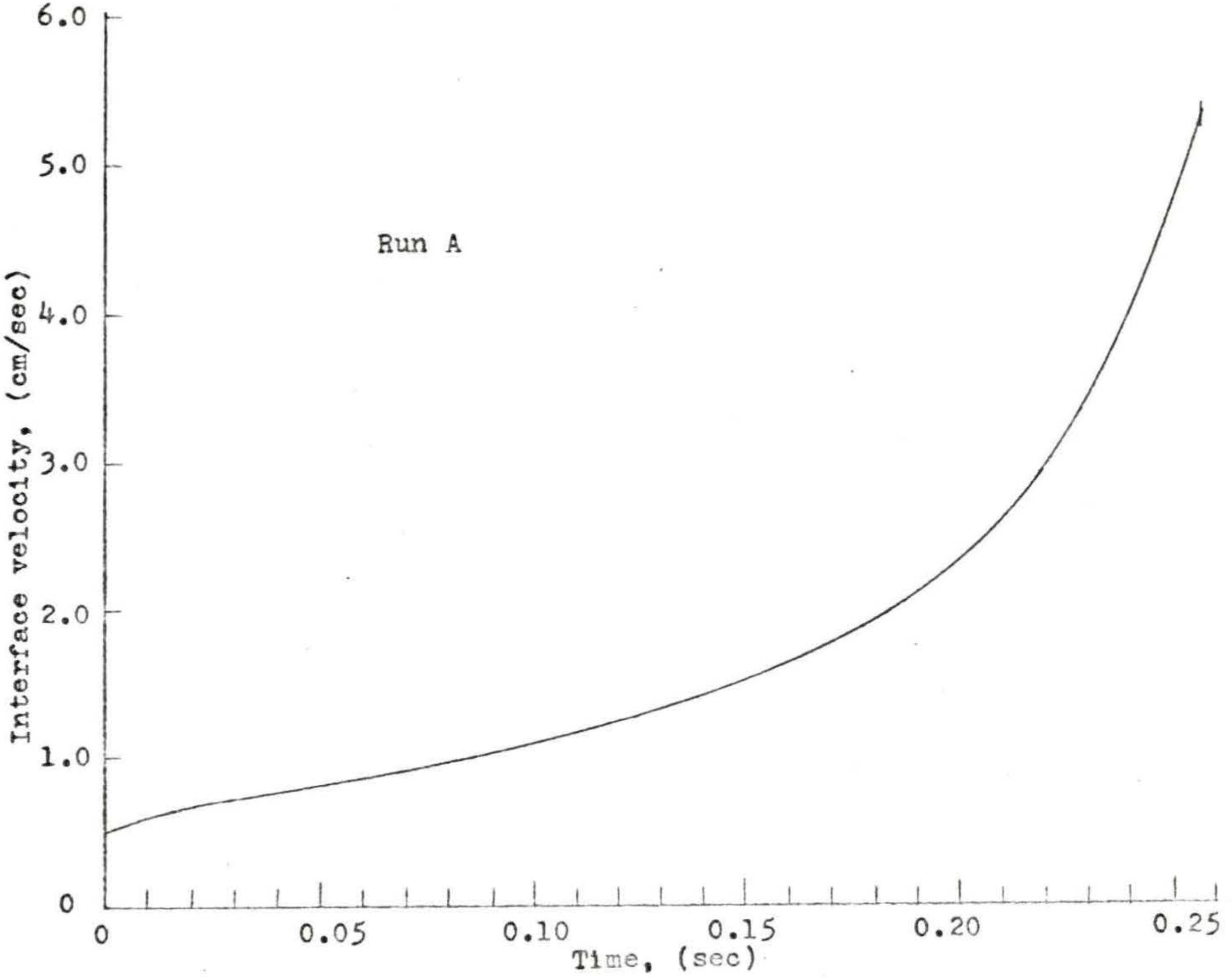


Figure 8A. Interface velocity.

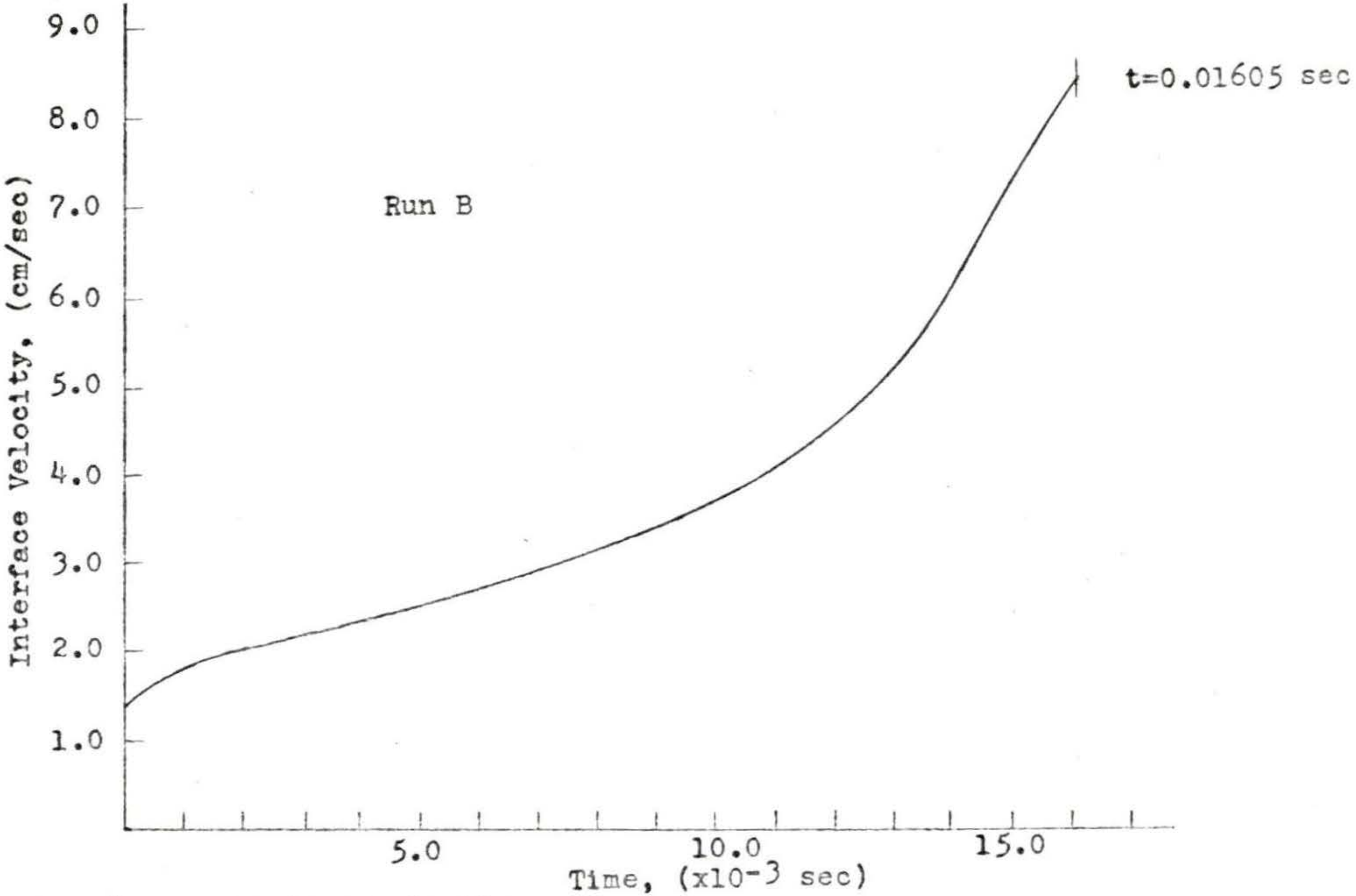


Figure 8B. Interface velocity.

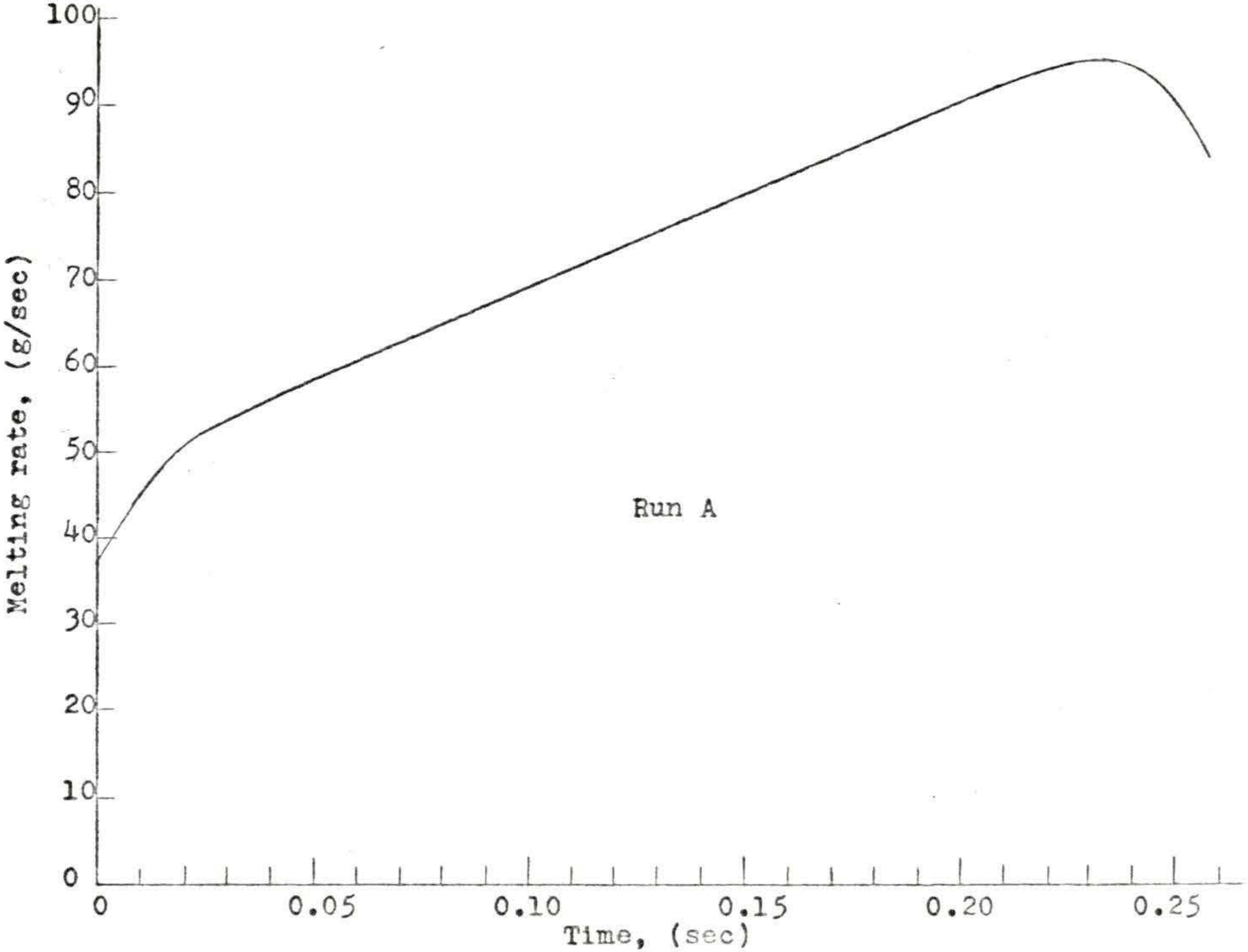


Figure 9A. Melting rate of coolant material.

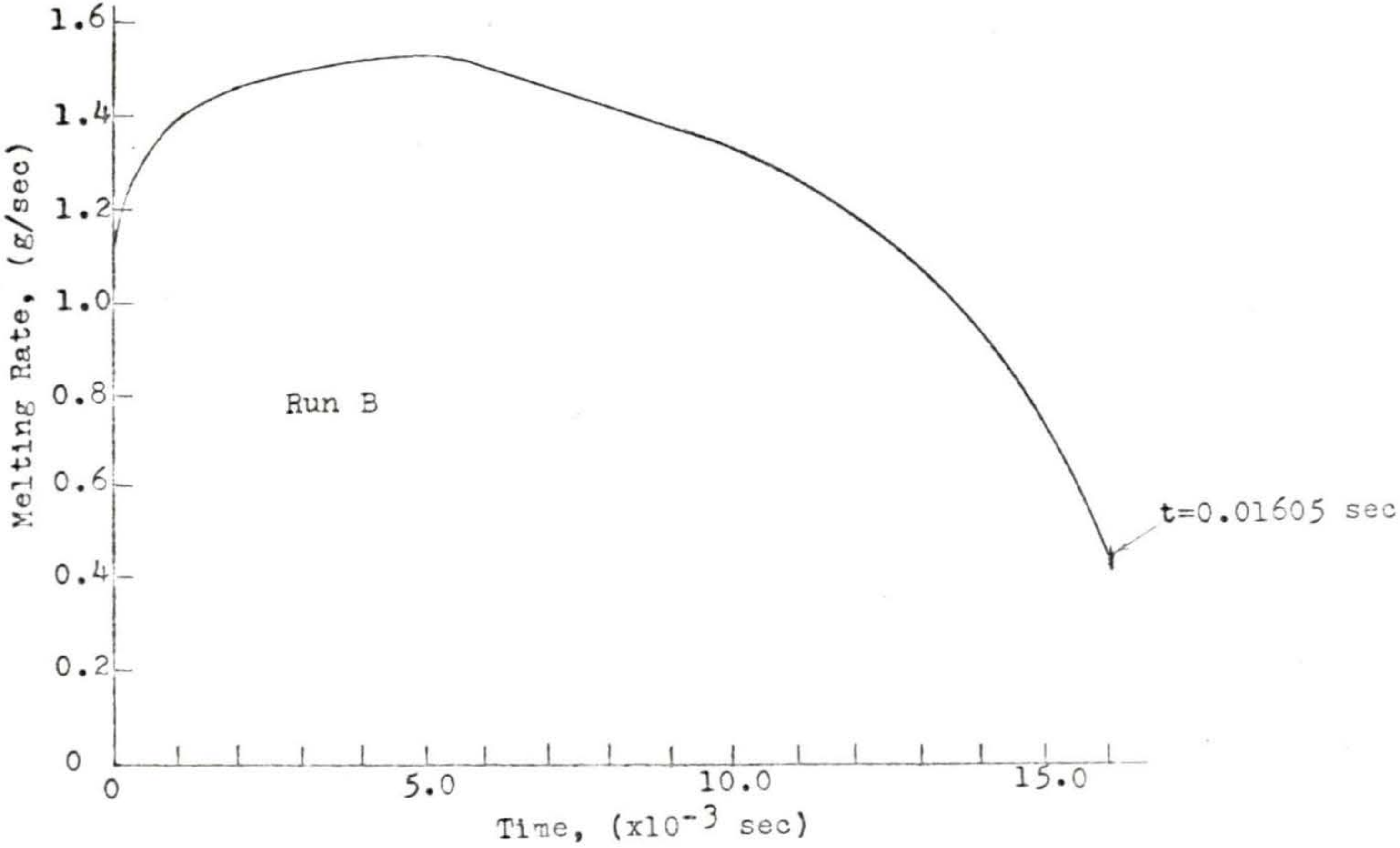


Figure 9B. Melting rate of coolant material.

APPENDIX B

Stress Analysis of a Thick-Walled Sphere

(The following approach is similar to that presented by Murphy (15, p. 114) for thick-walled cyclinders.)

Assume that the sphere is made up of spherical shells of infinitesimal thickness dr. Each of these shells is subjected to an internal radial stress and an external radial stress. From symmetry considerations, these stresses will be uniformly distributed on the surface of the shell. All stresses are assumed to be tensile, thus negative values will indicate that the stresses are compressive. From the freebody diagram of the hemispherical shell in Figure 10 and the equations of equilibrium one has

$$2\pi r\sigma_t dr + \pi r^2\sigma_r = \pi(r + dr)^2(\sigma_r + d\sigma_r) \qquad (109)$$

Neglecting second order differentials, this reduces to

$$\sigma_t = \sigma_r + \frac{r}{2}\frac{d\sigma_r}{dr} \qquad (110)$$

From Figure 11 the displacement of any Point A must be along a radial line if the stresses on the inside and outside of the spherical shell are uniformly distributed. From this fact the unit strain ϵ_t in the tangential direction due to a radial displacement U can be found as

$$\epsilon_t = \frac{2\pi(r + U) - 2\pi r}{2\pi r} = \frac{U}{r} \qquad (111)$$

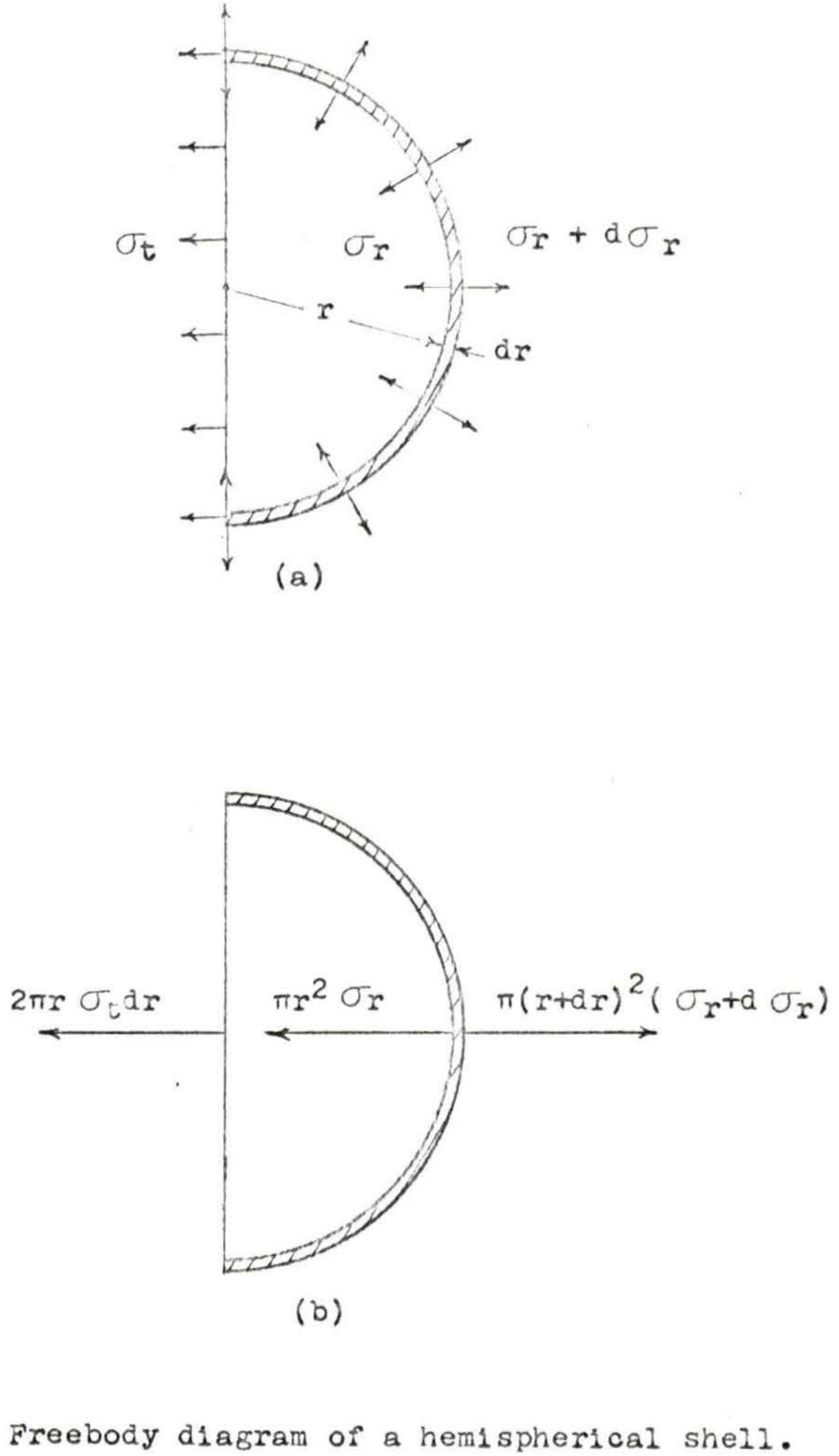


Figure 10. Freebody diagram of a hemispherical shell.

The average radial unit strain is seen to be

$$\epsilon_r = \frac{du}{dr} \tag{112}$$

It is also shown that for a triaxial system with direction u, v, and w

$$\sigma_u = \frac{E\left[\epsilon_u + M(-\epsilon_u + \epsilon_v + \epsilon_w)\right]}{1 - M - 2M^2} \tag{113}$$

where the material is isotropic and homogeneous and the proportional limit of neither normal nor sheer stress have been exceeded. The symbol E stands for the modulus of elasticity, and M is Poisson's ratio. For the use of a sphere the triaxial system consists of one radial and two tangential directions. Applying this to Equation 113 and substituting the expression for ϵ_t and ϵ_r from Equations 111 and 112 one can see that

$$\sigma_t = \frac{E}{1 - M - 2M^2}\left[\frac{u}{r} + M\frac{du}{dr}\right] \tag{114}$$

and

$$\sigma_r = \frac{E}{1 - M - 2M^2}\left[\frac{du}{dr} + M\left(-\frac{du}{dr} + \frac{2u}{r}\right)\right] \tag{115}$$

If Equations 115, 114, and 110, are combined and the common factors of E and $(1 - M - 2M^2)$ are canceled out, then

$$\frac{u}{r} + M\frac{du}{dr} = \frac{du}{dr} + M\left(-\frac{du}{dr} + 2\frac{u}{r}\right)$$

$$+ \frac{r}{2}\frac{d}{dr}\left[\frac{du}{dr} + M\left(-\frac{du}{dr} + 2\frac{u}{r}\right)\right] \tag{116}$$

This expression reduces to

$$\frac{r}{2}\frac{d^2u}{dr^2} + \frac{du}{dr} - \frac{u}{r} = 0 \qquad (116a)$$

If an assumed solution of the form

$$u = Cr^x \qquad (117)$$

is substituted into Equation 116a, it is found that $x = -2$ or $x = +1$. Thus one solution to Equation 116a has the form

$$u = Ar^{-2} + Br \qquad (118)$$

By using the boundary conditions of the problem the constants A and B can be evaluated. The boundary conditions are the radial the stresses at the inner and outer surfaces of the shell. The radial stress is given by Equation 115 with Equation 118 as

$$\sigma_r = \frac{E}{1 - M - 2M^2}\left[(2M - 1)\frac{2A}{r^3} + (1 + M)B\right] \qquad (119)$$

The boundary conditions for the inner and outer surfaces respectively are

$$\sigma_r(r = R_1) = - p_i \qquad (120)$$

and

$$\sigma_r(r = R_2) = - p_0 \qquad (121)$$

that is, the radial stress on a surface is a compressive stress equal in magnitude to the pressure exerted on the surface. With these conditions Equation 119 for the two surfaces becomes

$$- p_i = \frac{E}{1 - M - 2M^2}\left[(2M-1)\frac{2A}{{R_1}^3} + (1+M)B\right] \tag{122}$$

and

$$- p_0 = \frac{E}{1 - M - 2M^2}\left[(2M-1)\frac{2A}{{R_2}^3} + (1+M)B\right] \tag{123}$$

Using these two equations the solutions for the constants A and B can be shown to be

$$A = \frac{R_1^3 R_2^3 (p_0 - p_i)\ (1 - M - 2M^2)}{2E\ (2M - 1)\ (R_2^3 - R_1^3)} \tag{124}$$

and

$$B = \frac{(p_i R_1^3 - p_0 R_2^3)\ (1 - M - 2M^2)}{2E\ (1 + M)\ (R_2^3 - R_1^3)} \tag{125}$$

From these values of A and B the expressions for u, σ_r, and σ_t become

$$u = \frac{R_1^3 R_2^3 (p_0 - p_i)\ (1 + M) + 2(p_i R_1^3 - p_0 R_2^3\ (2M - 1)\ r^3}{2E\ (R_2^3 - R_1^3)\ r^2} \tag{126}$$

$$\sigma_t = \frac{p_i R_1^3 - p_0 R_2^3}{(R_2^3 - R_1^3)} - \frac{R_1^3 R_2^3 (p_0 - p_i)}{2r^3 (R_2^3 - R_1^3)} \tag{127}$$

$$\sigma_r = \frac{p_i R_1^3 - p_0 R_2^3}{(R_2^3 - R_1^3} + \frac{R_1^3 R_2^3 (p_0 - p_i)}{r^3 (R_2^3 - R_1^3} \tag{128}$$

These results agree with the results given for

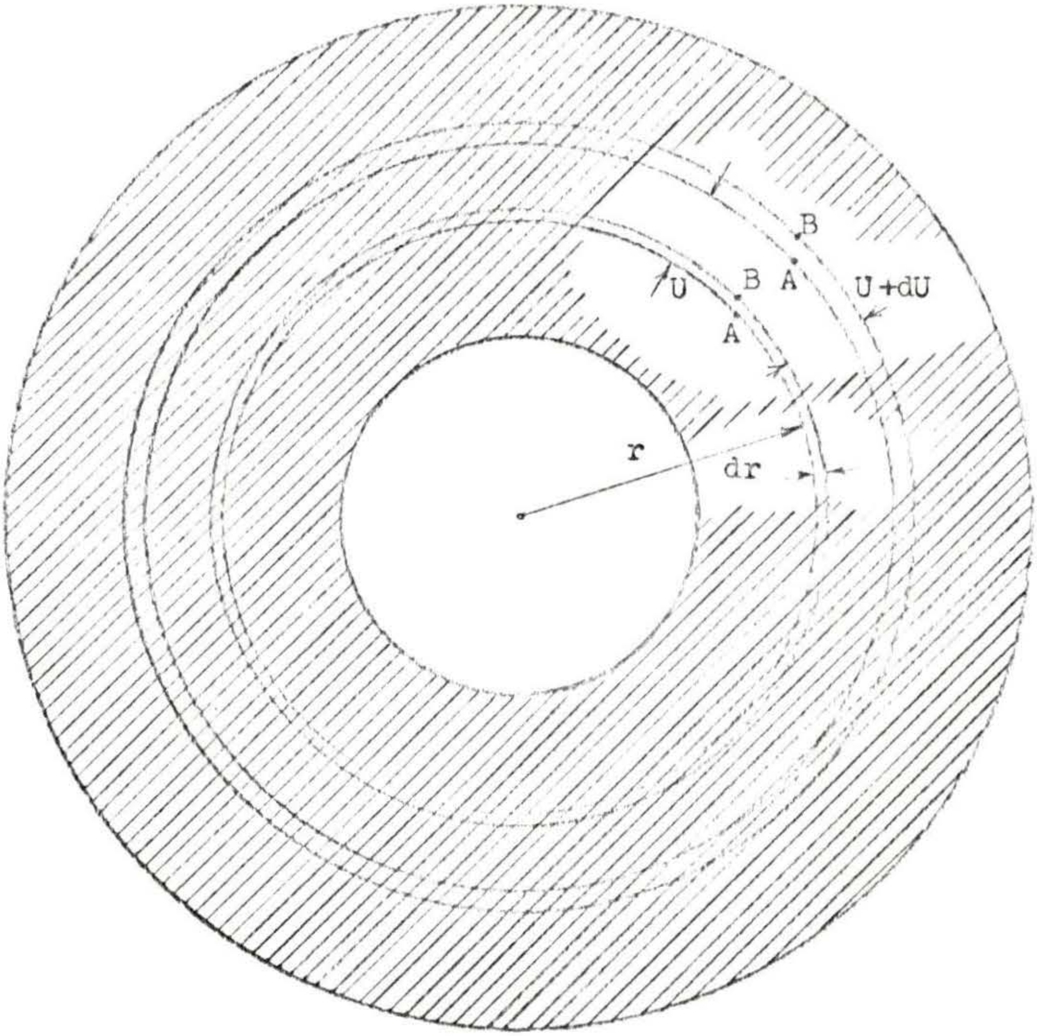


Figure 11. Displacement due to stress in a spherical shell.

various special cases by Roark (18, p. 232) and by Timoshenko and Goodier (22, p. 359).

Equations 109 through 119 hold equally well for solid spheres. The difference between a thick-walled shell and a solid sphere occurs in the boundary conditions and evaluation of the constants A and B. The boundary conditions at the outer surface are the same in both cases, but the boundary condition at the inside surface is replaced by the condition that the solution must be finite within the sphere. This condition implies that A must be zero. The expression for the displacement u thus becomes

$$u = Br \tag{129}$$

By evaluating Equation 129 at the outside surface the constant B is found to be

$$B = \frac{-p_0(2M - 1)}{E} \tag{130}$$

The resulting expressions for u, σ_r, and σ_t are

$$u = \frac{-p_0(2M - 1)}{E}r \tag{131}$$

$$\sigma_t = -p_0 \tag{132}$$

$$\sigma_r = -p_0 \tag{133}$$

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