The relationship between noncircular shape and plasma beta in toroidal MHD stability

by

Yasuhiro Baba

A Thesis Submitted to the
Graduate Faculty in Partial Fulfillment of the
Requirements for the Degree of
MASTER OF SCIENCE

Major: Nuclear Engineering

Iowa State University
Ames, Iowa

1986

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I. INTRODUCTION

We are utilizing nuclear energy. Essentially, nuclear energy consists of two types: one is fission energy and the other is fusion energy. However, right now, fusion energy is not available. To actualize fusion reactors using the principle known as magnetic confinement, there are two possibilities: one is a toroidal fusion reactor, which has no end out of which plasma can leak; the other is a magnetic mirror fusion reactor, which is open ended [1,2].

One type of toroidal fusion reactor, called tokamak [3], is the most possible to actualize, so far. In a tokamak, the most important plasma parameters are the plasma cross section, plasma current, plasma temperature, and plasma density. Also important is the value of "beta," which is the ratio of the plasma kinetic pressure to the magnetic pressure used to confine the plasma. Beta is affected by the plasma cross section. If a plasma cross section shape is noncircular, the beta value is higher than that of a circular-shaped plasma. Thus, concepts of toroidal fusion reactors have evolved, first from circular to noncircular cross section shape, then to elliptic and D shapes [1,2]. More recently, bean-shaped cross sections have been proposed [4]. All of these shapes show progressively higher beta values. The plasma current yields a poloidal magnetic field and toroidal magnetic coils yield the toroidal magnetic field in a tokamak system.

In this research, the relationship between a noncircular shape of a plasma cross section is focused on, especially a bean shape and the beta value.

The primary objectives of this research are:

1. To construct an improved, mathematically convenient plasma surface function for the bean-shape plasma cross section. The plasma surface function is a mathematical expression that defines the plasma surface.

2. To analyze the relationship between plasma current distribution and MHD instabilities for the bean-shape plasma.

3. To simplify the current density distribution function for the bean-shape plasma cross section.

4. To simplify the poloidal magnetic field distribution function according to the simplified current density distribution function.

5. To analyze the relationship between beta values and the poloidal magnetic field distribution function.

II. THEORY AND GOVERNING EQUATIONS

A. Basic Definitions

The value of beta, β, in a controlled thermonuclear reactor (CTR) is defined generally as

$$\beta = \frac{P}{B_o^2/2\mu_o} \qquad (1)$$

where P is the kinetic pressure of plasma, B_o is the magnetic field that confines plasma in a toroidal device, and μ_o is the permeability of the plasma in vacuum. Thus, a poloidal beta, β_P, can be defined as

$$\beta_P = \frac{P}{B_P^2/2\mu_o} \qquad (2)$$

where B_P is the poloidal magnetic field to confine plasma. This field is

$$B_P = \frac{I_P}{2\pi\ a_e} \qquad (3)$$

where I_P is the plasma current to produce the poloidal magnetic field and a_e is the effective minor radius of plasma. Similarly, a toroidal beta, β_t, can be defined as

$$\beta_t = \frac{P}{B_t^2/2\mu_o} \qquad (4)$$

where B_t is the toroidal magnetic field.

B. Plasma Cross Section Shape Calculation

For the bean-shape cross section of plasma, the following equations have been suggested to define its surface [4,5,6]:

$$X(\theta) = \tilde{X} + \rho \cos t$$

$$Z(\theta) = E \rho \sin t \qquad (5)$$

The terms in this discussion are identified in Fig. 1. In equation (5), $\rho = 1 + B \cos \theta$, $t = C \sin \theta$, and $0 \leq \theta \leq 2\pi$. To quantify the effects of indentation, $i \equiv d/2a$, the parameters $\tilde{X}$, E, and C in these equations can be varied so as to keep the elongation, b/a, and the aspect ratio, R/a, constant.

Instead of using the above equation, the following equations are proposed here:

$$X_o(\theta) = v(\cos \theta - 1) + u \cos \theta \sqrt{k^2 - \frac{4\theta^2}{\pi^2}}$$

$$Z_o(\theta) = v \sin \theta + u \sin \theta \sqrt{k^2 - \frac{4\theta^2}{\pi^2}} \qquad (6)$$

Equation (6) is in effect in the convex part of the curve in Fig. 2. In addition, the following equations

$$X_i(\theta) = v(\cos \theta - 1) - u \cos \theta \sqrt{k^2 - \frac{4\theta^2}{2}}$$

$$Z_i(\theta) = v \sin \theta - u \sin \theta \sqrt{k^2 - \frac{4\theta^2}{2}} \qquad (7)$$

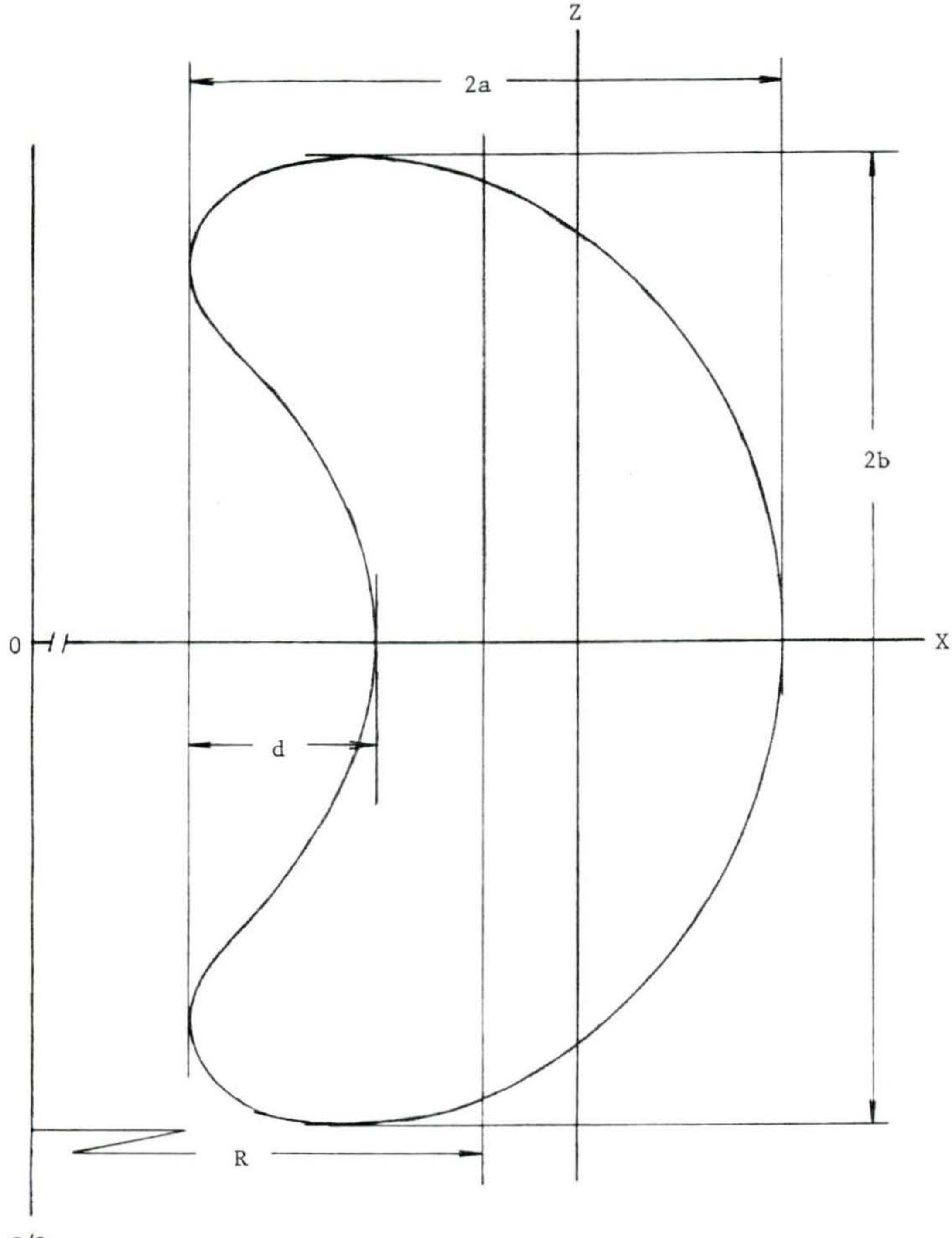

Figure 1. Bean-shaped plasma cross section

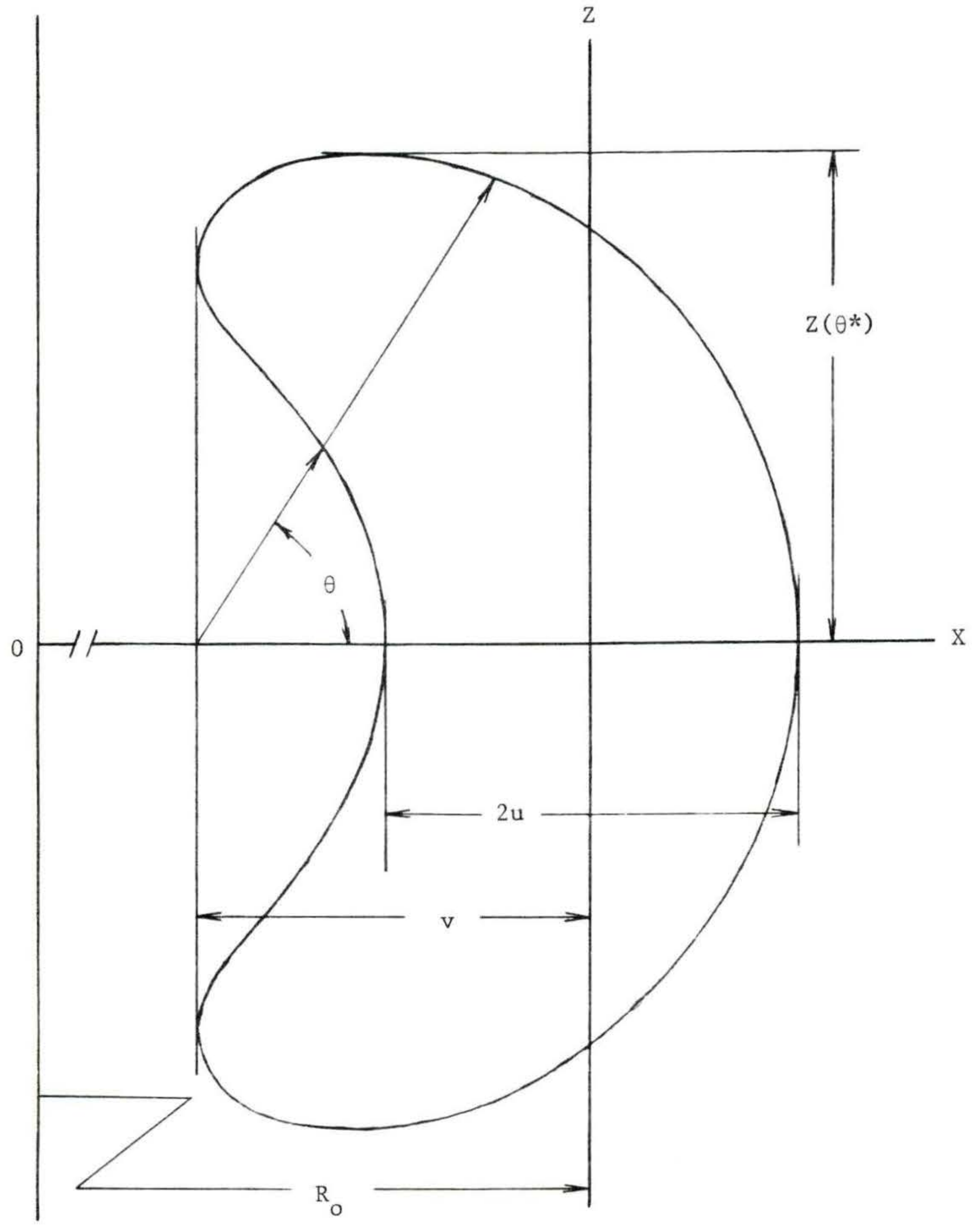

Figure 2. Bean-shaped plasma cross section

are in effect to describe the concave part of the curve in Fig. 2. In equations (6) and (7)

$$0 \le k \le 1 \tag{8}$$

If k is equal to 1, equations (6) and (7) describe the outer surface of the plasma. If k is less than 1, equations (6) and (7) may be used to define "inner profiles" of the plasma. The variable, θ, is in the range

$$-\frac{k\pi}{2} \le \theta \le \frac{k\pi}{2} \tag{9}$$

The relationships between the coordinates shown in Fig. 1 and the coordinates used in equations (6) and (7) are as follows:

$$i \equiv \frac{d}{2a} = \frac{v - u}{u + v} \tag{10}$$

$$a = \frac{1}{2}(u + v) \tag{11}$$

$$b = Z(\theta^*) \tag{12}$$

The θ^* satisfies the following equations:

$$\tan \theta^* = \frac{\pi^2}{4\theta^*}\frac{v}{u}\sqrt{1 - \frac{4\theta^{*2}}{2}} + \frac{\pi^2}{4\theta^*} - \theta^* \tag{13}$$

$$R = R_o - \frac{u - v}{2} \tag{14}$$

Equations (6) and (14) depend on the assumption that the magnetic axis of the plasma, that is, the point of highest magnetic field, lies at the origin. This is also the centroid of the toroidal plasma current distribution. This assumption is understood from the experimental data shown in Fig. 3, which fit this assumption very well [4].

The idea of the equations (6) and (7) is as follows: In a function of $Z = f(X)$, which has the condition of $|Z| \leq \pi v/2$, if Z axis is mapped on the following equation:

$$X = -v + \sqrt{v^2 - Z^2} \tag{15}$$

we can get the equation of $Z = g^{o}f(X)$. From this mapping procedure, equations (6) and (7) are given.

C. Cross Sectional Area Element of Bean-shaped Plasma

To figure out the poloidal magnetic field of bean-shaped plasma, the cross-sectional area element is calculated. From equations (6) and (7), the cross-sectional area element, S(k), is

$$S(k) = \int_{o}^{\frac{k\pi}{2}} \left[r_o^2 - r_i^2\right] d\theta \tag{16}$$

where

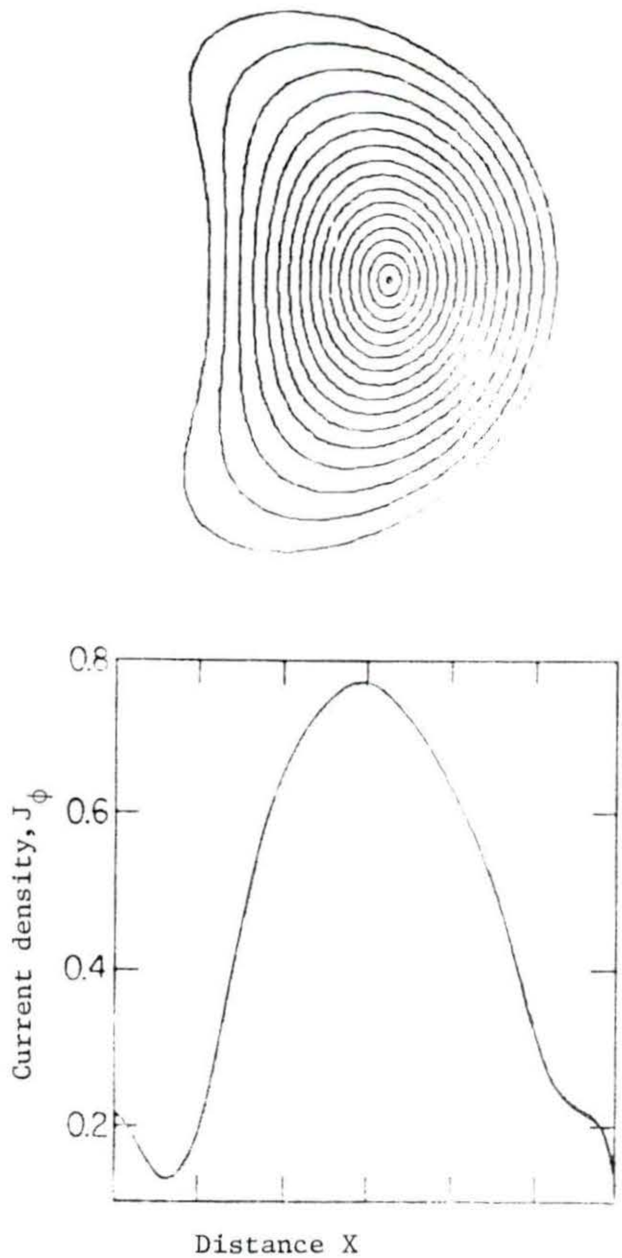

Figure 3a. Current density distribution and plasma cross section ($\langle\beta\rangle \sim 0.1\%$)

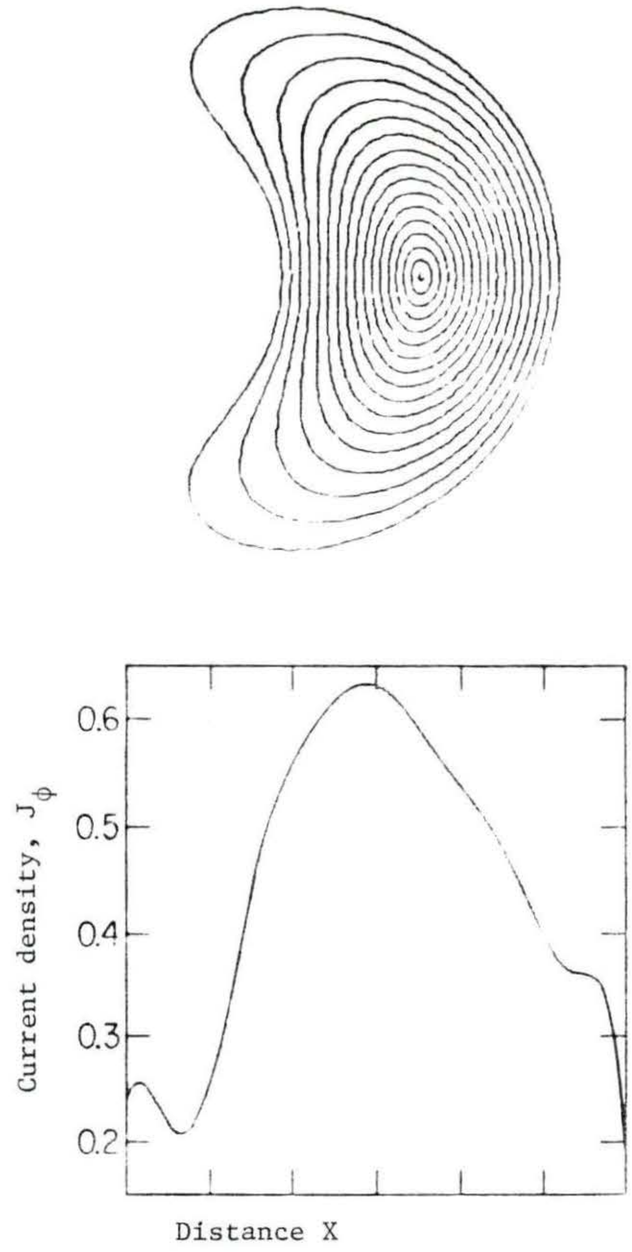

Figure 3b. Current density distribution and plasma cross section (<β> ~ 0.1%)

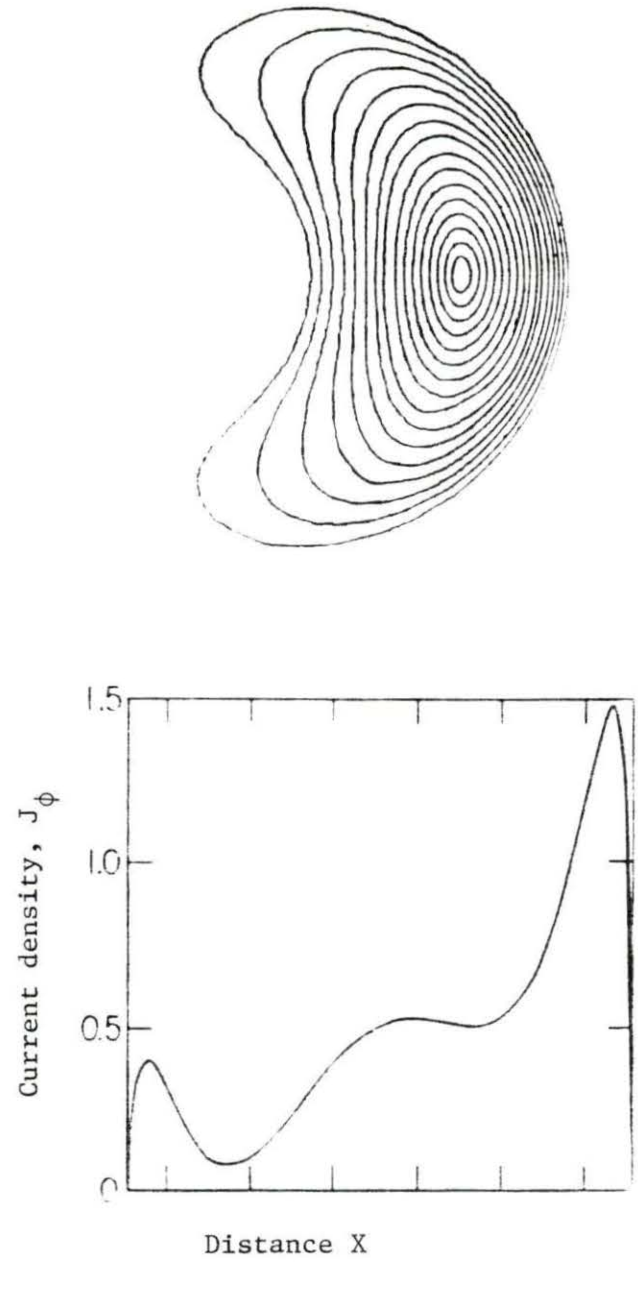

Figure 3c. Current density distribution and plasma cross section ($\langle\beta\rangle \sim 10\%$)

$$r_o = r_o(\theta) = \sqrt{\{X_o(\theta) + v\}^2 + \{Z_o(\theta)\}^2}$$
$$r_i = r_i(\theta) = \sqrt{\{X_i(\theta) + v\}^2 + \{Z_i(\theta)\}^2} \tag{17}$$

Inserting equation (17) into equation (16) gives

$$S(k) = 4\ u\ v \int_o^{\frac{k\pi}{2}} \sqrt{k^2 - \frac{4\theta^2}{\pi^2}}\ d\theta \tag{18}$$

The variable, θ, can be transformed as

$$\theta = \frac{k\pi}{2} \sin \phi \tag{19}$$

Inserting equation (19) into equation (18) gives

$$S(k) = 4\ u\ v \int_o^{\frac{\pi}{2}} k \cos \phi \frac{k\pi}{2} \cos \phi\ d\phi$$
$$= \pi\ u\ v\ k^2\ [\phi + \frac{1}{2} \sin 2\phi]_o^{\frac{\pi}{2}} = \frac{\pi^2\ u\ v\ k^2}{2} \tag{20}$$

D. Plasma Current Density Distribution

The plasma current density distribution figures are given in Fig. 3 as experimental data. Usually, plasma current density distributions are approximated as

$$J_\phi = J_o [1 - \{\frac{r(k,\phi)}{r_s(1,\phi)}\}^{\alpha}]^{\beta} \tag{21}$$

where α and β are constants, $r(k,\phi)$ is the distance between an arbitrary point and the origin, $r_s(1,\phi)$ is the distance between the origin and the surface and ϕ is the angle between the X axis and $\vec{r}$. Usually, $\alpha = \beta = 2$ are chosen [7].

The plasma current at a cross section is given by:

$$I_P = \int_S J \, dS \tag{22}$$

where S is the cross sectional area of the plasma, J is current density, and ds is area element.

The poloidal magnetic field distribution about the origin can be given as

$$B_P(r(k,\phi)) = \frac{\mu_o}{2\pi r(k,\phi)} \int_S J_\phi \, dS$$

$$= \frac{J_o \mu_o}{2\pi r(k,\phi)} \int_S [1 - (\frac{r(k,\phi)}{r_s(1,\phi)})^\alpha]^\beta \, dS \tag{23}$$

The current density distribution shown in Figure 4 is described by the equation:

$$J_\phi = J_o \, [1 - \{\frac{r(k,\phi)}{r_s(1,\phi)}\}^2]^2 \tag{24}$$

However, $r(k,\phi)/r_s(1,\phi)$ is actually equal to k, as may be verified from equation (7). Then, equation (24) can be expressed as

$$J_\phi = J_o[1 - k^2]^2 \tag{25}$$

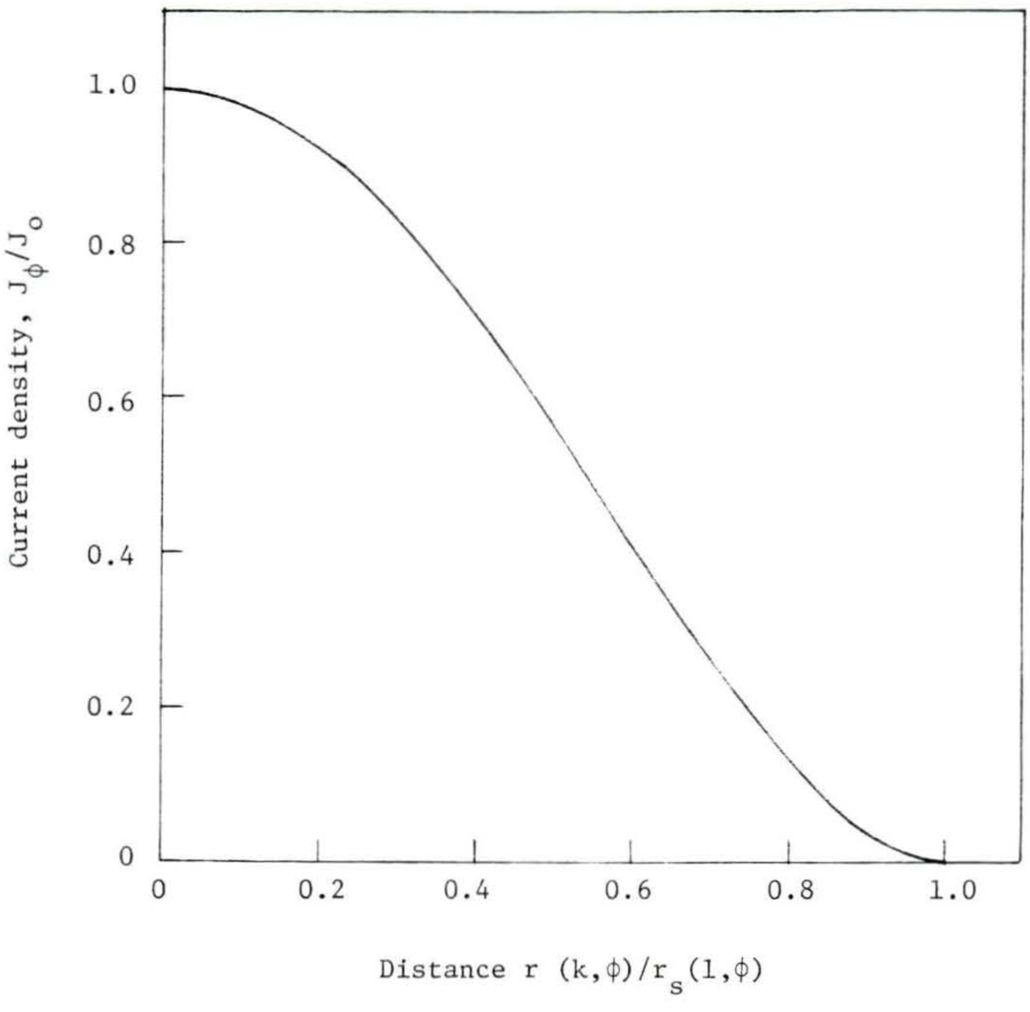

Figure 4. Current density distribution

According to the above, the poloidal magnetic field distribution is

$$B_p(k) = \frac{\mu_o}{2\pi k} \int_o^k J_\phi \, ds = \frac{\mu_o}{2\pi k} \int_o^k J_o(1 - k'^2)^2 \, \pi^2 uvk'dk'$$

$$= \frac{\pi\mu_o J_o uv}{12} [k^5 - 3k^3 + 3k] \qquad (26)$$

The above expression is shown in Figure 5.

From the above expression, the curve which is made by the isodynamic lines is given in Figure 6.

E. Toroidal Magnetic Field

In the toroidal magnetic field of the bean-shaped plasma, there is a tendency for electrons to move along the magnetic lines, so that the electric fields which are produced by the gradients of the magnetic fields are canceled by the movement of electrons along the rotational transformed magnetic lines. Thus, the maximum toroidal magnetic fields occur at the intersection of the convex curve and X axis, and at the two vortexes.

F. Safety Factor

The definition of safety factor, q, is the ratio of the winding number of magnetic force lines around the major axis to the winding number of magnetic force lines around the minor axis. In general, the value of q differs according to the distance from the minor axis in a cross section of plasma. The safety factor is expressed as

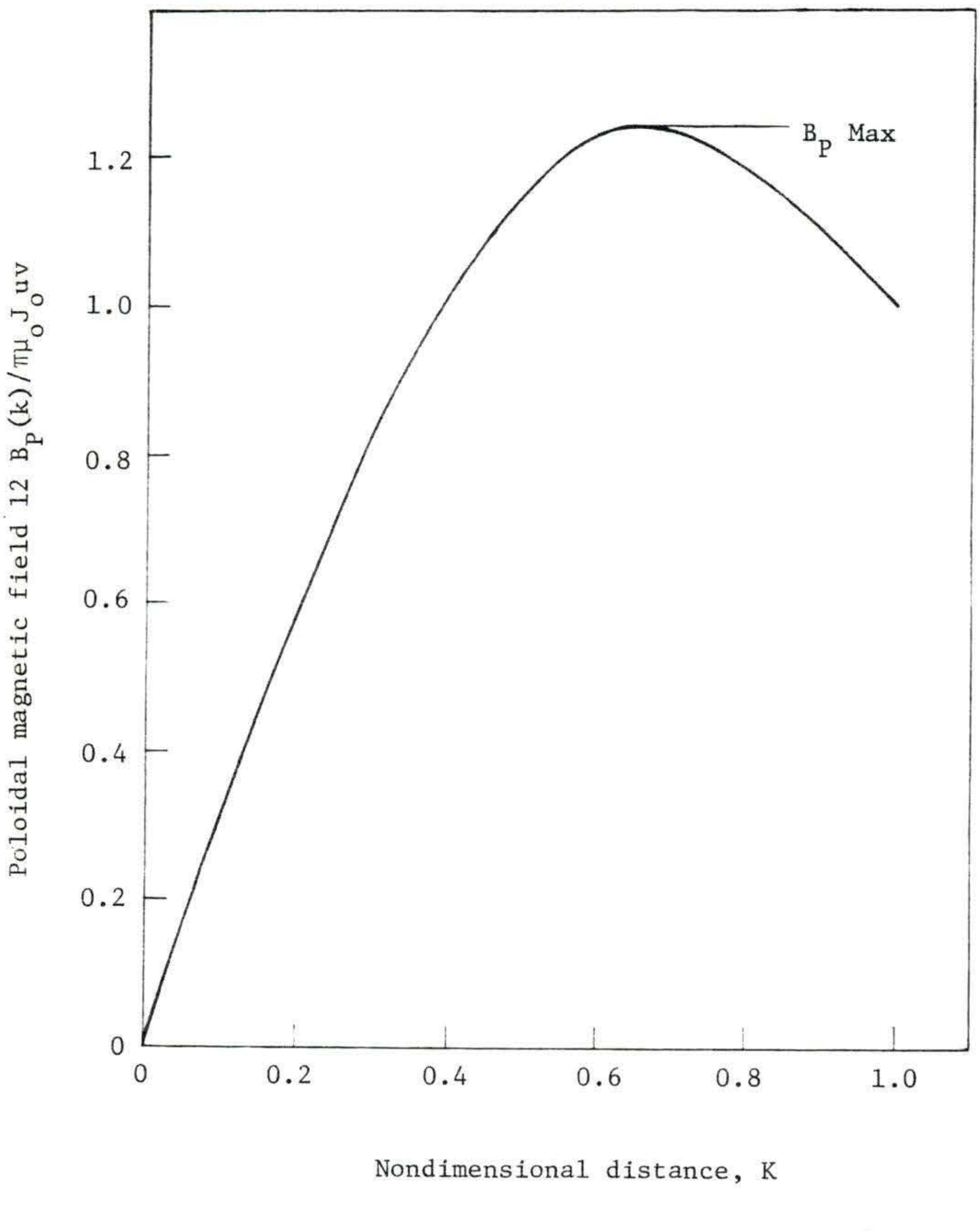

Figure 5. Poloidal magnetic field distribution

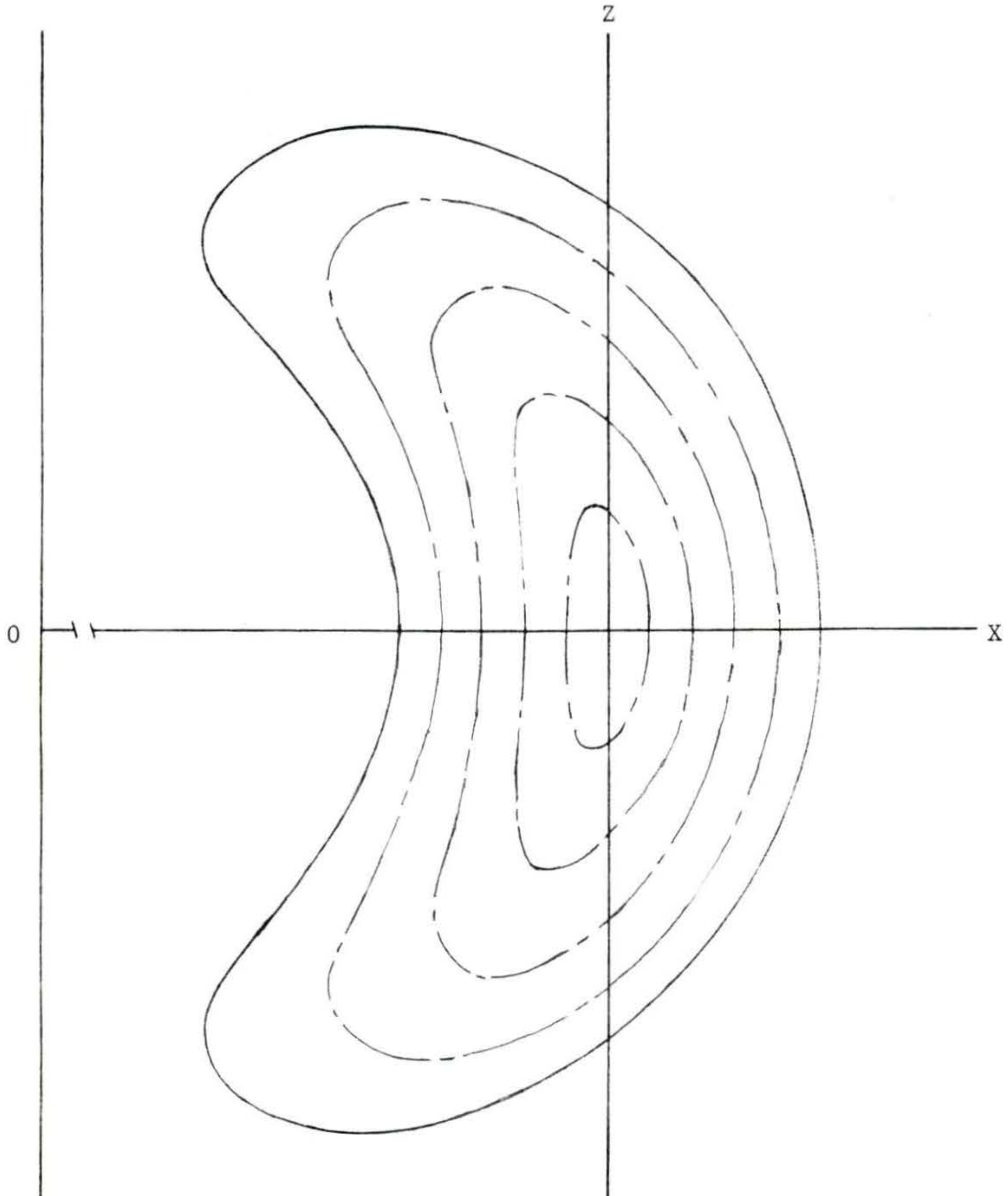

Figure 6. Isodynamic lines

$$q = \oint_c \frac{B_t}{B_P} \frac{d\ell}{2\pi R_o} \tag{27}$$

where the integration contour, c, is along the magnetic force line around the minor axis. Near the minor axis, which corresponds to the magnetic axis of the poloidal magnetic field, B_P is almost maximum according to Figure 5, however; therefore, the integration contour is almost zero. Then, q is given by taking the limit as r goes to zero. On the other hand, at the plasma surface, q is given by:

$$q = \frac{1}{B_P(r_s)} \oint_c B_t(r_s) \frac{d\ell}{2\pi R_o} \tag{28}$$

where the integration contour is along the plasma surface. From the previous calculation, the value of poloidal magnetic field at the plasma surface is constant.

If the toroidal magnetic field at the surface is almost constant, which is the case when the major radius is much larger than the minor radius, equation (28) can be expressed as

$$q(r_s) = \frac{B_t(r_s)}{B_P(r_s)} \oint_c \frac{d\ell}{2\pi R_o} \tag{29}$$

Equation (29) can be calculated as

$$q(r_s) = \frac{B_t(r_s)}{B_P(r_s)} \frac{2\pi a_e}{2\pi R_o} = \frac{B_t(r_s)}{B_P(r_s)} \frac{a_e}{R_o} \tag{30}$$

where a_e, the effective radius of plasma, is defined as

$$a_e = \sqrt{\frac{S}{\pi}} = \sqrt{\frac{\pi uv}{2}} \tag{31}$$

Here, S is the cross sectional area of plasma at the surface, as in equation (19).

The effective aspect ratio, A_e, is expressed as

$$A_e = \frac{R_o}{a_e} \tag{32}$$

Therefore, the safety factor at the plasma surface is

$$q(r_s) = \frac{B_t(r_s)}{B_P(r_s)} \frac{1}{A_e} \tag{33}$$

The relationship between the poloidal beta, β_P, and the toroidal beta, β_t, is given by the combination of equations (2), (4), and (33). It is

$$\beta_t = \beta_P \frac{1}{q^2(r_s)} \frac{1}{A_e^2} \tag{34}$$

To confine plasma, it is desired that the value of the toroidal beta be as large as possible. From equation (34), it follows that the small values of $q(r_s)$ and A_e are to be chosen.

III. DISCUSSION AND APPLICATION

A. Properties of MHD Instabilities

There are many kinds of MHD instabilities [8]. Their properties are strongly related to the poloidal magnetic field. For example, the tearing mode instability takes place on the poloidal magnetic field. The process of the tearing mode instability is as follows:

1. Local magnetic fields perturb the poloidal magnetic field.
2. They produce new magnetic axes which are different from the magnetic axis of the poloidal magnetic field.
3. They split off the poloidal magnetic field.

As another example, there is the ballooning mode instability. Here, plasma is expanded in the direction from the magnetic axis of the poloidal magnetic field to the outside. This expansion is increased by centrifugal force.

B. Poloidal Magnetic Field Analysis

As exhibited in Figure 3, there are several maximum values in each plasma current density distribution. This is a problem. According to the distribution function, some portions of the plasma behave like paramagnetic particles and the others like diamagnetic particles. Especially in Figure 3c, there are three maximum values in the distribution, which can be explained as arising from:

1. Centrifugal force
2. Skin effect

Particularly, if a very large current is applied to plasma, the inductance increases. If there are more than two maximal values, the conductivity increases slowly. The above processes affect the tearing mode instability. In the condition of Figure 3c, there is a high possibility for new magnetic axes to appear.

The most desirable current distribution has only one maximum value which exists at or near the magnetic axis. Even though the tearing mode instability does not occur, the centrifugal force is always present. There is a tendency for the ballooning mode instability to increase if charged particles in the plasma have large energy.

To avoid the above situation, the magnetic field along the major axis is always applied; however, the above instability still takes place at a high enough plasma energy (temperature). If constant magnetic fields along the major axis are applied, there is no improvement because the application affects only the magnitude of the poloidal magnetic field. Magnetic fields which increase along the direction from the plasma surface to magnetic axis are better; however, this is not easy to do. If the magnitude of an applied field is not sufficient, some maximal magnetic fields might still remain.

It follows that, from the magnetic axis to the plasma surface, the poloidal magnetic field, $B_p(k)$ must be a monotone increasing function. In other words,

$$\frac{d}{dk} B_p(k) > 0 \; ; \qquad 0 \leq k \leq 1 \tag{35}$$

If a point of inflection exists in the function after adding an applied field, then at the point of inflection we get

$$\frac{d}{dk} B_P(k)\Big|_{k=i} = 0 \ ; \qquad 0 < k < 1 \ \text{ and } \ 0 < i < 1 \ .$$

where i is the point of inflection

It is possible for charged particles to stay near such a point, creating the possibility that the magnetic field at that point becomes unstable.

C. Toroidal Magnetic Field Analysis

The minimum toroidal magnetic field exists at the center of the inner surface of the bean-shaped plasma and the maximum toroidal magnetic field exists directly opposite on the outer surface. From the mid-plane, the toroidal magnetic field distribution, $B_t(k)$, is a monotone increasing function and along the mid-plane, it is a constant:

$$\frac{d}{dk} B_t(k) \geq 0 \ ; \qquad 0 < k < 1 \ .$$

This situation is not desirable. If the toroidal magnetic field distribution is considered, the distribution along Z axis is always constant for any value of k; however, the distribution perpendicular to the Z axis has the maximum and minimum values. To reduce the difference between the maximum value of $B_t(k)$ and the minimum value of $B_t(k)$, a large value of aspect ratio is chosen.

D. Current Density Distribution Analysis

The following types of current density distribution are discussed here.

1. Constant current
2. Skin effect current
3. Current with inside conductor
4. Current with outside conductor

Their poloidal magnetic field distributions are also calculated.

1. Constant current

Constant current means that the current density is constant. The expression is as follows:

$$J_\phi = J_o = \text{constant} \; ; \qquad 0 \le k \le 1 \, . \tag{36}$$

In this case, using equation (20) gives

$$B_P(k) = \frac{\mu_o}{2\pi k} \int_o^k J_o \, \pi^2 \, uvk' \, dk'$$

$$= \frac{\pi \mu_o \, u \, v \, J_o}{2k} \int_o^k k' \, dk' = \frac{\pi \mu_o \, J_o \, u \, v}{4} \tag{37}$$

Equations (36) and (37) are shown in Figure 7. According to Figure 7, the poloidal magnetic field distribution has the property that

$$\frac{d}{dk} B_P(k) > 0 \, .$$

Thus, $B_P(k)$ is stable.

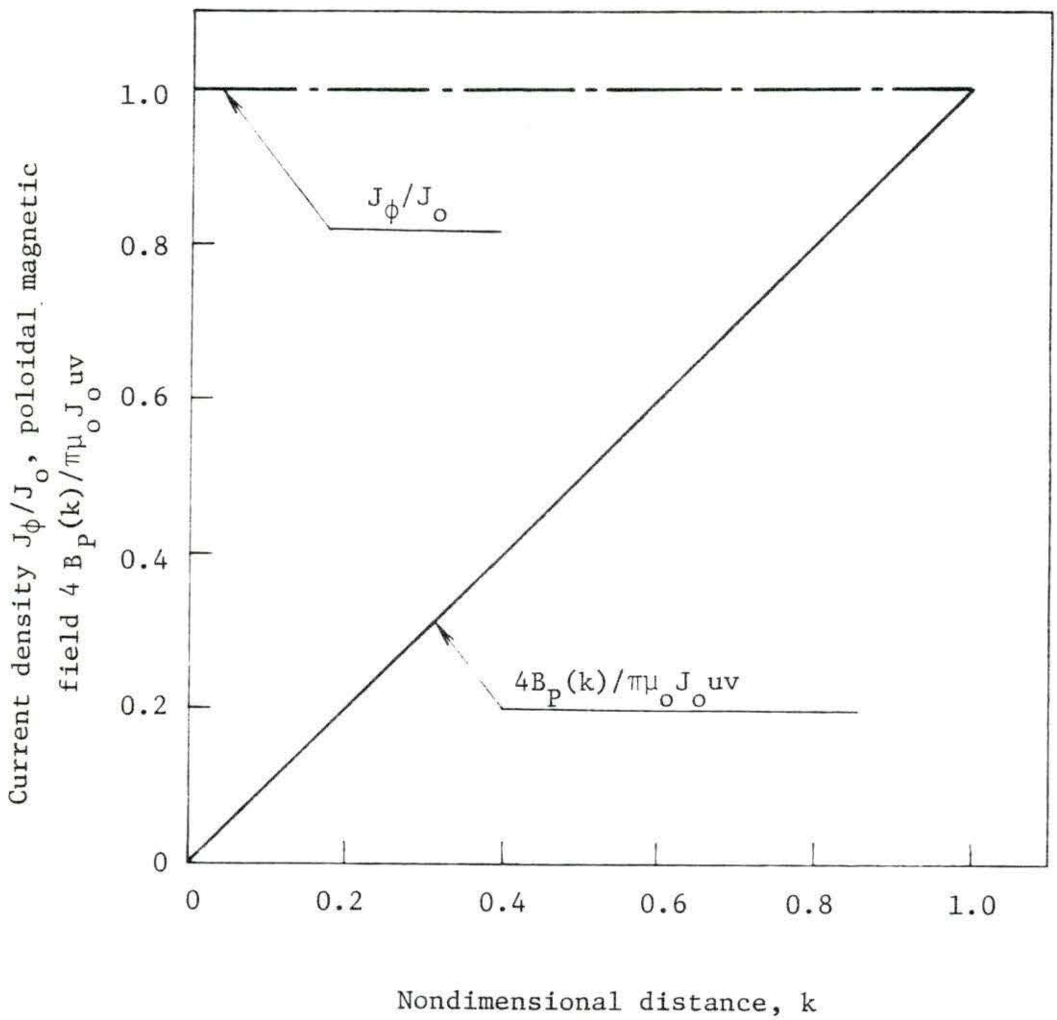

Figure 7. Current density distribution and poloidal magnetic field distribution for constant current density distribution

2. Skin effect current

Skin effect current is caused by high frequency wave current. If high frequency wave current is applied to a conductor, inside inductance increases and most of the current density is concentrated on the surface of the conductor. The mathematical expression for skin effect current is:

$$J_\phi = J_o \, \delta(k) \; ; \qquad \delta(k) = \begin{cases} 1 : k = 1 \\ 0 : k \neq 1 \end{cases} \tag{38}$$

where $\delta(k)$ is delta function.

Instead of using a delta function, it may be approximated by higher order function. Actually, some current density does exist in the range of $0 < k < 1$ so that the delta function does not fit the distribution perfectly in any case. Moreover, with a delta function, there is no magnetic field in the range of $0 < k < 1$ and the magnetic field exists only in the range of $k \geq 1$. Thus, a high order power function is more realistic. Such a function is expressed as

$$Y = X^n \; ; \qquad 0 \leq X \leq 1. \tag{39}$$

Applying equation (39) to equation (38) gives

$$J_\phi = J_o \, k^n \tag{40}$$

In the case of n = 4, equation (40) is

$$J_\phi = J_o \, k^4 \tag{41}$$

Combining equations (20) and (41) gives

$$B_P(k) = \frac{\mu_o}{2\pi k} \int_o^k J_o \pi^2 uvk'^5 \, dk'$$

$$= \frac{J_o \mu_o \pi uv}{2k} \int_o^k k'^5 \, dk' = \frac{J_o \mu_o \pi uv}{12} k^5 \qquad (42)$$

Equations (41) and (42) are shown in Figure 8. According to Figure 8, the poloidal magnetic field has the property of

$$\frac{d}{dk} B(k) > 0$$

Thus, $B_P(k)$ is stable and skin effect current is better than constant current because near the plasma surface, the gradient of the poloidal magnetic field for skin effect current is much larger than that of constant current.

3. Current with inside conductor

This system is called internal ring system. A conductor is placed inside the plasma; a system which has more than one internal ring is called multipole. In this section, only one internal ring system is discussed.

Let the following equations be set up:

$$J_c = J_o = \text{constant} ; \qquad 0 \leq k \leq \alpha < 1 \qquad (43)$$

$$J_P = J_o \frac{1}{(1-\alpha)^2} [-k^2 + 2\alpha k - 2\alpha + 1] ; \qquad 0 < \alpha \leq k \leq 1. \qquad (44)$$

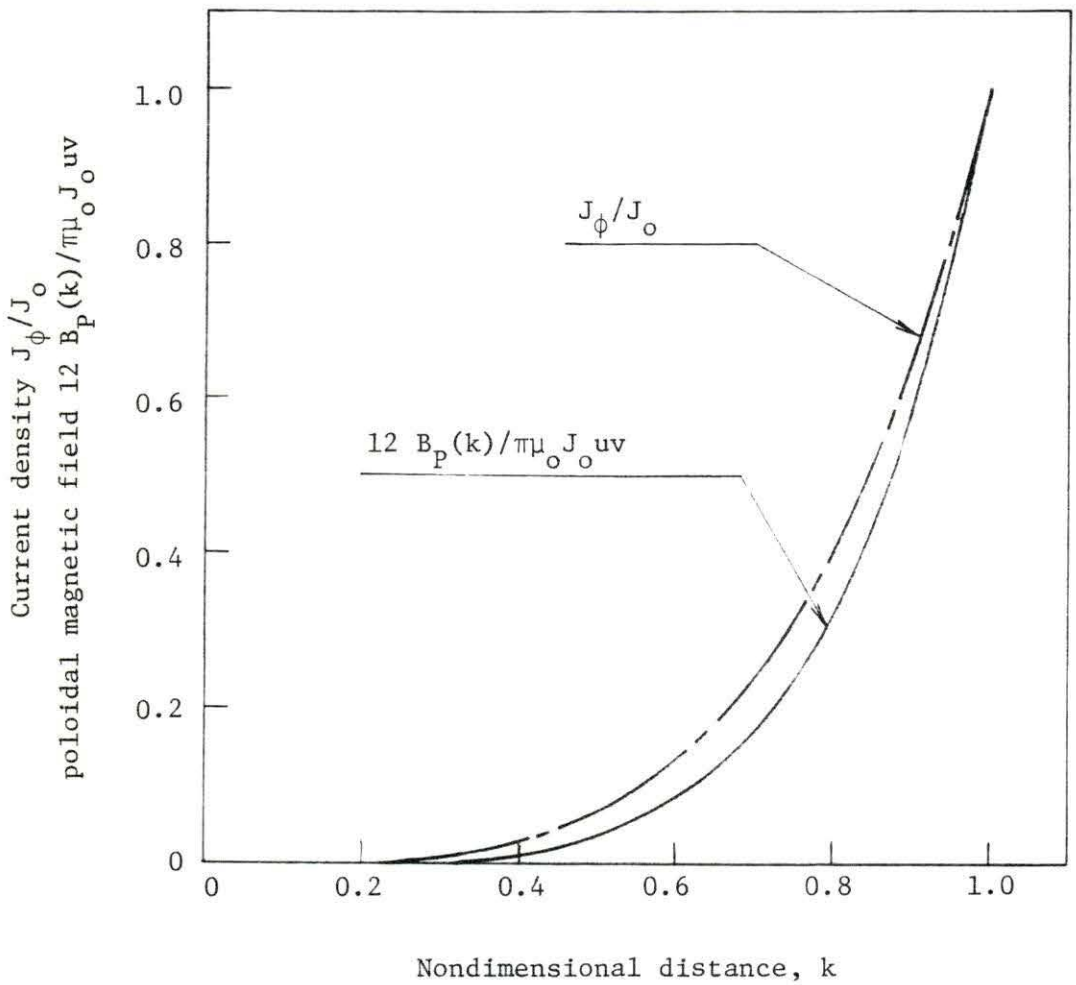

Figure 8. Current density distribution and poloidal magnetic field distribution for skin effect current

where α is dimensionless conductor thickness, J_c is the conductor current density and J_p is the plasma current density. The assumptions are that the maximum value of J, J_{PM}, takes place at $k = \alpha$; at this place, $J_{PM} = J_c = J_o$. At the plasma surface, the plasma current density is equal to zero.

The poloidal magnetic field is the sum of the magnetic field which is produced by a conductor and that which is produced by plasma. Moreover, the magnetic field which is produced by plasma current does not exist in the conductor.

The magnetic field of the conductor is

$$B_c(k) = \frac{\mu_o}{2\pi k} \int_o^k J_o \pi^2 uvk' \, dk'$$

$$= \frac{J_o \mu_o \pi uv}{4} k \, ; \qquad (0 \leq k \leq \alpha) \qquad (45)$$

$$B_c(k) = \frac{\mu_o}{2\pi k} \int_o^k J_o \pi^2 uvk' \, dk'$$

$$= \frac{J_o \mu_o \pi uv\alpha^2}{4k} \, ; \qquad \alpha \leq k \leq 1. \qquad (46)$$

Similarly, the magnetic field of plasma is

$$B_{PP}(k) = \frac{\mu_o}{2\pi k} \int_\alpha^k \pi^2 uvk' \, J_o \left[\frac{-k'^2 + 2\alpha k' - 2\alpha + 1}{(1-\alpha)^2}\right] dk'$$

$$= \frac{\pi J_o \mu_o uv}{2(1-\alpha)^2} \left[- \frac{k^3}{4} + \frac{2}{3} \alpha k^2 + \frac{1-2\alpha}{2} k - \frac{\alpha^2}{12k}(5\alpha^2 - 12\alpha + 6)\right] \qquad (47)$$

Thus, the compound magnetic field is given by

$$B_P(k) = B_c(k) = \frac{J_o\mu_o\pi uv}{4} k \ ; \qquad 0 \leq k \leq \alpha \tag{48}$$

$$B_P(k) = B_c(k) + B_{PP}(k)$$

$$= \frac{J_o\mu_o\pi uv\alpha^2}{4k} + \frac{J_o\pi\mu_o uv}{2(1-\alpha)^2}\left[-\frac{k^3}{4} + \frac{2}{3}\alpha k^2 + \frac{1-2\alpha}{2}k - \frac{\alpha^2}{12k}(5\alpha - 12\alpha + 6)\right] \tag{49}$$

Equations (48) and (49) are plotted in Figure 9. In the range of $0 \leq k \leq \alpha$, there is no plasma, and the gradient of $B_P(k)$ is

$$\frac{d}{dk} B_P(k) < 0 \ ; \qquad \alpha \leq k \leq 1.$$

Thus, the plasma has a paramagnetic character by the existence of the conductor. Experimentally, this system gives good results as concerning stability; however, the problem is to have a conductor in the plasma. If very high temperature is applied to the conductor, it will melt down. If this material problem can be solved, this system is the best way to get a stabilized poloidal magnetic field.

4. Current with outside conductor

This system is to surround plasma with conductor. Even if high current is applied to the conductor, the magnetic field which is produced by the conductor never exists in the plasma. Thus, the magnetic

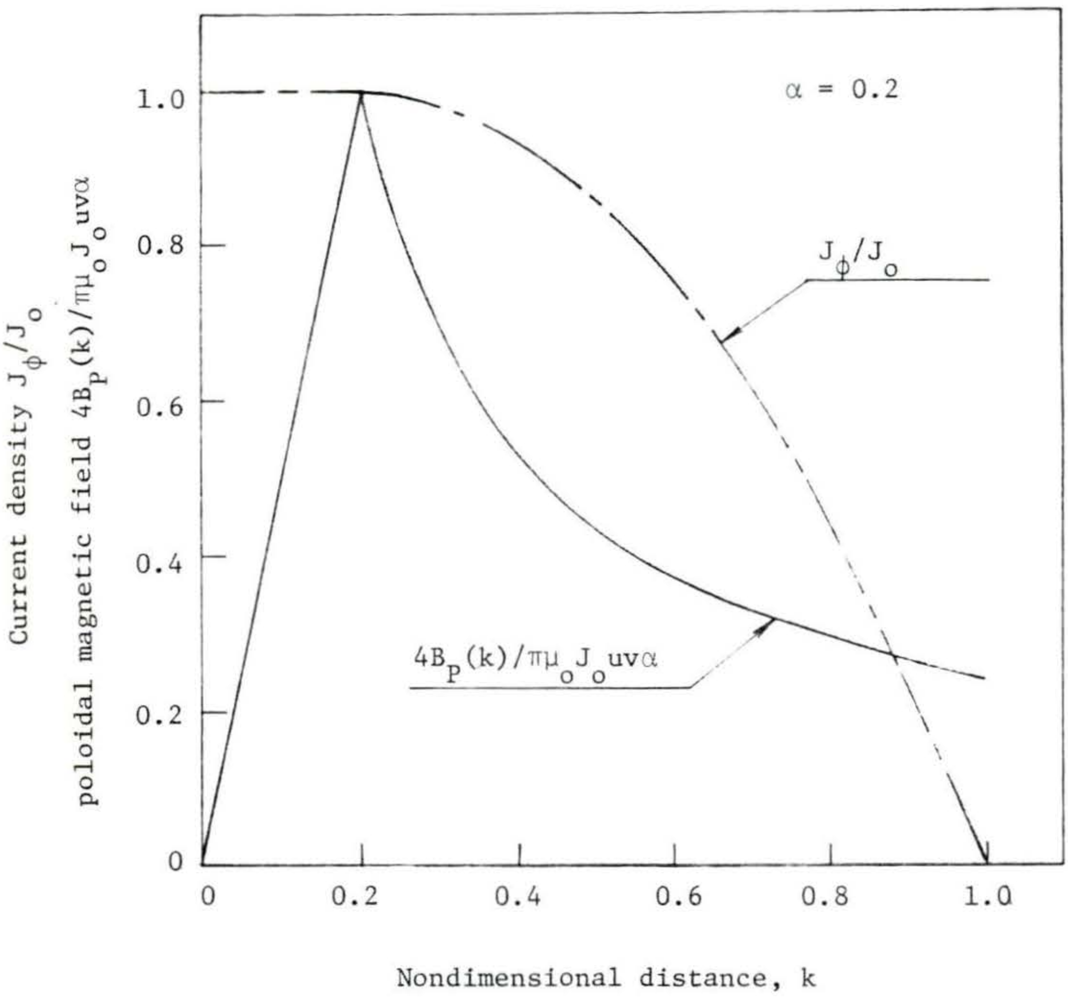

Figure 9. Current density distribution and poloidal magnetic field distribution for the current with a conductor

field distribution in plasma is the same with or without the magnetic field of the conductor.

E. Effect of Conductor

If plasma is surrounded with a conductor, the position of plasma is always balanced. If the magnetic axis of plasma slips off against the center of the conductor, current flows in the toroidal direction at the inner wall of the conductor, and this current yields a vertical magnetic field, keeping the plasma in balance automatically. Therefore, an external conductor is effective for stabilizing kink mode instability.

F. Effect of Indentation

The characteristic of a bean-shaped plasma is to have an indentation. In an indented plasma, the relative elongation increases as compared to D-shaped, elliptic-shaped, and circular-shaped plasma.

The poloidal beta depends only on plasma current; however, the toroidal beta depends not only on the safety factor but also on the aspect ratio. If the poloidal beta and safety factor are fixed, a small aspect ratio is desirable for the toroidal beta according to equation (34). The indentation affects the aspect ratio, that is, the aspect ratio gets smaller after applying the indentation according to equations (29) and (34). The indentation thus permits a high toroidal beta; however, a critical indentation will exist according to Figure 3c, and at this point cannot be increased further.

G. Beta Value Analysis

The equations of the plasma beta are shown in equations (1) to (4). They arise from the fundamental equations:

$$\vec{J} \times \vec{B} = \nabla P \tag{50}$$

$$\nabla \times \vec{B} = \mu_o \vec{J} \tag{51}$$

$$\nabla \cdot \vec{B} = 0 \tag{52}$$

Equation (50) is the balance of forces, equations (51) and (52) are Maxwell's equations.

Using the above equations and eliminating $\vec{J}$ gives

$$\nabla P = \frac{1}{\mu_o} \{(\vec{B}\cdot\nabla)\vec{B} - \nabla(\frac{B^2}{2})\} \tag{53}$$

In the above, the term $(\vec{B}\cdot\nabla)\vec{B}$ is based on curvature of magnetic force lines. If the curvature can be neglected, equation (53) can be expressed as

$$\nabla(P + \frac{B^2}{2\mu_o}) = 0 \tag{54}$$

Thus,

$$P + \frac{B_i^2}{2\mu_o} = \frac{B_o^2}{2\mu_o} \tag{55}$$

where B_i is the magnetic field in plasma and B_o is the magnetic field without plasma $(P = 0)$.

According to the definition of plasma beta, equation (55) is

$$\beta = \frac{P}{B_o^2/2\mu_o} = \frac{B_o^2 - B_i^2}{B_o^2} = 1 - (\frac{B_i}{B_o})^2 \qquad (56)$$

Considering the poloidal beta, we can neglect curvature because consideration is focused on the plasma cross section. Therefore, equation (56) is expressed as

$$\beta_P(k) = 1 - \{\frac{B_P(k)}{B_P(1)}\}^2 \qquad (57)$$

The average poloidal beta, $<\beta_P>$, is

$$<\beta_P> = \int_o^1 \beta_P \, dk = \int_o^1 [1 - \frac{B_P^2(k)}{B_P^2(1)}] \, dk \qquad (58)$$

For example, if plasma current distribution is given by equation (25), the poloidal beta distribution is

$$\beta_P = 1 - \frac{(k^5 - 3k^3 + 3k)^2}{1} = - k^{10} + 6k^8 - 15k^6 + 18k^4 - 9k^2 + 1$$

and the average poloidal beta is

$$<\beta_P> = \int_o^1 \beta_P \, dk = \int_o^1 [- k^{10} + 6k^8 + 15k^6 + 18k^4 - 9k^2 + 1] dk = \frac{38}{1155} \simeq 0.0329$$

For constant plasma current, equation (37) gives

$$\beta_P = 1 - k^2$$

and the average poloidal beta is

$$<\beta_P> = \int_o^1 \beta_P \, dk \int_o^1 [1 - k^2] dk = \frac{2}{3} \simeq 0.667$$

Roughly speaking, standard tokamak's circular-shaped cross section) have beta values which are almost equal to 0.01 with the current density distribution as given by equation (25).

It follows that a bean-shaped plasma cross section permits higher beta values compared to a circular cross section.

IV. SUMMARY AND CONCLUSION

From the preceding analysis and the discussion of the relationship between beta value and bean-shaped cross section in toroidal MHD stability, the following main conclusions are made:

1. Achieving a higher beta value depends strongly on the MHD stability and plasma shape. That is, more poloidal magnetic field analysis must be required. So far, the highest beta value has been achieved by bean-shaped cross sections experimentally.

2. For higher beta value, plasma current is very important. That is, it is very important to control plasma current or to put coils around the fusion reactor vacuum vessel, so that the plasma current distribution has only one maximal value near the magnetic axis.

3. From the theoretical analysis, the beta value of the bean-shaped cross section is higher than other toroidal systems.

So far, the poloidal magnetic field distribution produced by the plasma current has been considered. Actually, the plasma cross section is controlled by outer poloidal field coils. In this research, the effect of the poloidal magnetic field produced by outer poloidal field coils has been neglected. The combination of the poloidal magnetic field which is produced by the plasma current and the poloidal magnetic field which is produced by the outer poloidal field coils is very complicated to analyze. In the future, such an analysis must nevertheless be made. An external poloidal field could produce much higher beta values and perhaps stabilize the plasma under all conditions.

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VI. ACKNOWLEDGMENTS

The author would like to acknowledge Drs. B. M. Ma, B. I. Spinrad, and D. K. Finnemore who helped me to prepare for this thesis.